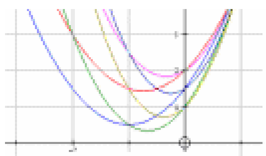
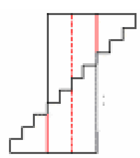
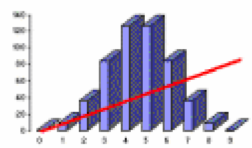
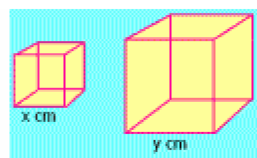
Risps: Rich Starting Points for A Level Core Mathematics

A Collection of Forty Open-ended Investigative Activities for the A Level Pure Mathematics Classroom

by Jonny Griffiths

Second Edition – April 2007

http://www.risps.net

http://www.jonny-griffiths.net

Foreword

Welcome to this collection of 'risps', and I hope you will soon be at home with this light-hearted acronym. My further and primary hope is that these activities will enrich the lives of mathematics students in lots of A Level classrooms. This will only happen, however, if they are seen as useful by teachers, who are likely to have only a few minutes to judge this for themselves. So **Part One** of this ebook immediately gives you the unadorned risps together with their Teacher Notes, indexed by topic, with no distractions.

Part Two tells the story of how these activities came about, and offers a philosophy as to how they might best be used. This can be read during a time of reflection (if busy teachers can manufacture such a thing) and will hopefully suggest ways to improve the writing and practice of risps old and new in the light of classroom experience.

So without more ado…

Part One: Contents

List of Risps by Topic

AS Level Core Mathematics

A2 Level Core Mathematics

7. Algebra 2

8. Trigonometry 2

9. Parametric Equations

10. Integration 3: Differential Equations

11. Vectors

Generic/Miscellaneous Risp Ideas

Risp 1: Triangle Number Differences

Pick two whole numbers between 1 and 10 inclusive,

and call them a and b.

Say that T_n is the nth triangle number.
Find T_a and T_b.

What is the difference between T_a and T_b?
Is this a prime number?

When is the difference between two triangle numbers a prime number?

When is the difference between two square numbers a prime number?
Between two cubes?

Risp 1: Triangle Number Differences

<u>Teacher notes</u>

Suggested Use: to introduce/consolidate/revise ideas of proof

Skills included:
to be able to construct and present a mathematical argument
appropriate use of logical deduction

My students began by playing with triangle numbers and their differences, forming conjectures, and testing them out. Once they had some results, the question arose, how best to organise these? A table seemed a good idea.

	1	3	6	10	15	21	28	36	45	55
1	0	2	5	9	14	20	27	35	44	54
3		0	3	7	12	18	25	33	42	52
6			0	4	9	15	22	30	39	49
10				0	5	11	18	26	35	45
15					0	6	13	21	30	40
21						0	7	15	24	34
28							0	8	17	27
36								0	9	19
45									0	10
55										0

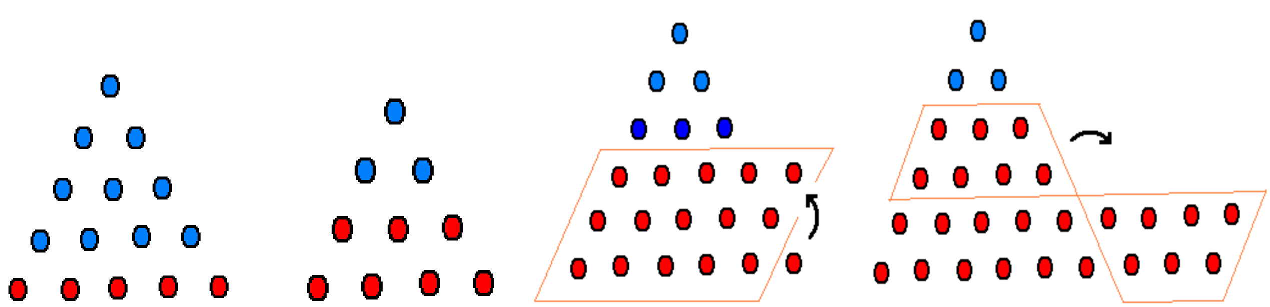
There was plenty to notice here. Some students noted that the primes fell close to the main diagonal.

"The only chance for $T_a - T_b$ to be prime is if a - b is 1 or 2."

This risp lends itself to visual as well as algebraic proof.

$T_a - T_b = a(a + 1)/2 - b(b + 1)/2 = (a^2 + a - b^2 - b)/2 = (a - b)(a + b + 1)/2$

So if a - b = 1 or 2, this could be prime. If a - b is 3, then a + b + 1 is even, so this can't be prime, and so on. Or there is this proof-without-words:



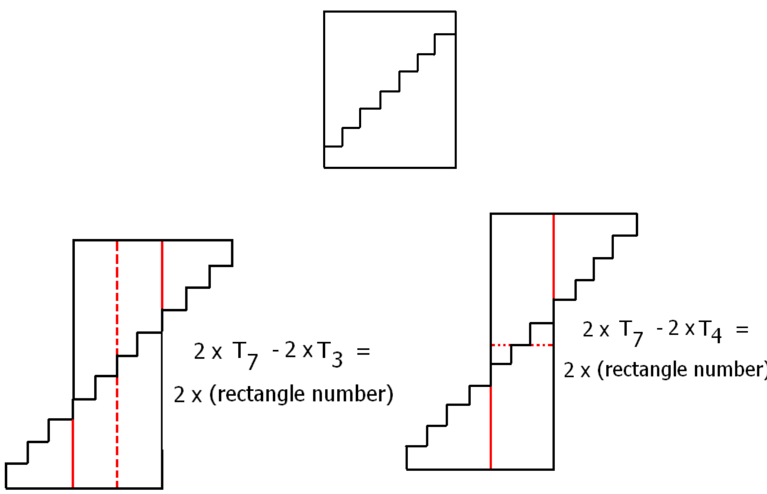
Students will hopefully go on to say that while $2 = T_2 - T_1$, and $3 = T_3 - T_2$, every other prime can be written in exactly two ways as the difference of two triangle numbers, since $p = T_p - T_{p-1}$, and $p = T_{(p+1)/2} - T_{(p-3)/2}$.

This risp occurred to me whilst on holiday in Barcelona.

Monument to Francesc Macia, Place Catalunya

by Josep Maria Subirachs

This suggested to me another proof without words:



I am grateful to Ian Short for suggesting the extension to the difference between two squares/cubes. Bernard Murphy helpfully pointed out that the difference of two fourth powers factorises neatly to show that it is never prime. Richard Peters gave me the excellent thought that calling the two triangle numbers T_r and T_{r+k} simplifies the discussion over when their difference is prime, since:

$$T_{r+k} - T_r = \frac{1}{2}(r + k)(r + k + 1) - \frac{1}{2}r(r+1) = \frac{1}{2}k(2r + k + 1).$$

Risp 2: Sequence Tiles

Arrange these tiles to define a sequence in as many different ways as you can.

$U_n =$	n	$)$	^{n}
1	$($	$\div$	$-$

How do these different sequences behave?

What happens if we add in the following tiles:

U_{n-1}	U_{n-2}

Are there other tiles you can add in to get other types of sequence?

Risp 2: Sequence Tiles

Teacher notes

Suggested Use: to consolidate/revise sequences/series

Skills included:
defining a sequence: the nth term in terms of n, and in terms of previous terms
identifying convergent, divergent, increasing,
decreasing, oscillating and periodic sequences

My students looked at these tiles and wondered what to do. They began to place them together in different ways, some defining legal sequences, and some not. Words like “divergent”, "convergent", “bounded” and “periodic” came into play, and during lively discussion students were able to say “I don’t know what ‘divergent’ means,” without being made to feel self-conscious. My class began to stick down tiles onto a poster to summarise our work. There was plenty of debate about what counted as convergent/divergent. Students asked if they could use spare tiles from other groups in their formulae, which I allowed.

I was able to draw up a table on the OHP of sequences we had found:

	Divergent	Convergent	Periodic	Other
Always increasing				
Always decreasing				
Oscillating				
"Flat"				

I threw out this challenge: “Can we fill the gaps using these tiles only?”

	Divergent	Convergent	Periodic	Other
Always increasing	n^n	$(n-1)/n$	NA	NA
Always decreasing	$-n$	$1/n$	NA	NA
Oscillating	$(-n)^n$	$(-1)^n/n$	$(-1)^n$	?
"Flat"	NA	$u_n = u_{n-1}$	NA	NA

Can we have an always-increasing yet bounded sequence that is not convergent? This activity raises some deep questions. There are lots of extensions. What tile would I need to add to fill the ? gap above? How about a 'last digit of' tile? Would [last digit of n] fill the gap? How about [last digit of n^n]?

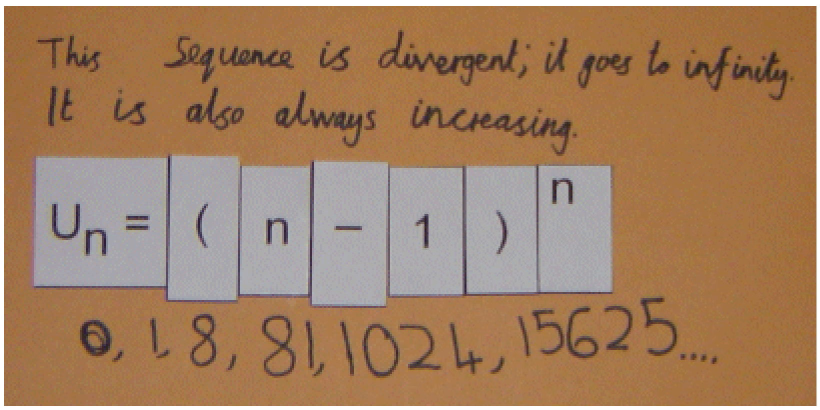
(I am grateful to Bernard Murphy for pointing out that (n-1)/n uses two ns, which is a little cheeky.)

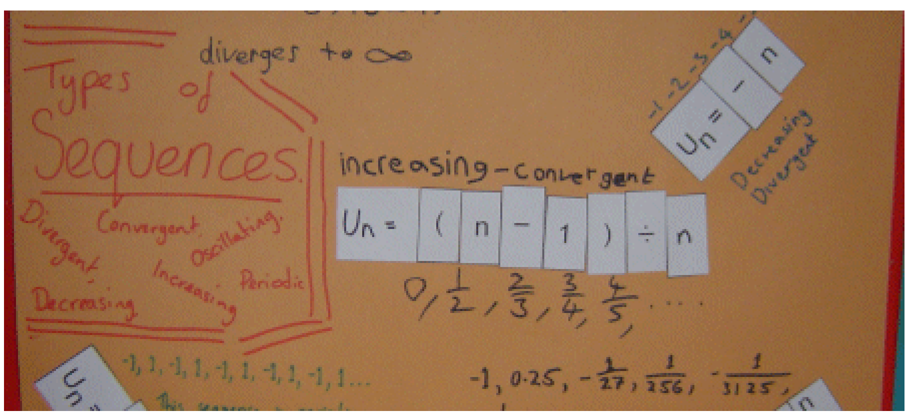
Adding in the u_{n-1} and u_{n-2} tiles creates some interesting sequences.

$u_n = 1/(u_{n-1}u_{n-2})$ is periodic, period 3, and $u_n = u_{n-2}/u_{n-1}$ is periodic, period 6.

The ? gap can be filled by u_n = nth digit after the decimal point in π.

Parts of our poster looked like this:





Risp 3: Brackets Out, Brackets In

Pick three different integers between -4 and 4 inclusive.

(0 is not allowed!)

Replace the squares below
with your three numbers in some order. (No repeats!)

$$(x+\square)(\square x+\square)$$

How many different orders are there?

Write down all these expressions, then...
multiply them all out, then...
add all the results together.

Now take this sum: can you factorise it?

Compare notes with your colleagues
once you have tried to do this.
Do you notice anything?

Does it matter what the starting list of numbers is?
Can you make any conjectures?
Can you prove these?

Risp 3: Brackets Out, Brackets In

Teacher notes

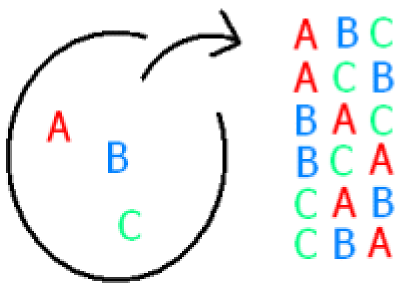
Suggested Use: to introduce/consolidate/revise simple expanding/factorising

Skills included:

general use of algebra, including expanding brackets, and factorising a quadratic
proof and mathematical argument

This risp is hot off the press, tried for the first time yesterday (does that make this a 'nouveau riche' starting point?)

Everyone who tries to write their own materials for their classroom develops a style. I very often use this idea of choosing a number of objects and then mixing them up in all possible ways within an expression or equation.



This generates a family of problems, and if you can find a property that links the results, you have a chance to set students plenty of practice at a particular skill whilst asking them to discover this property, or 'mini-theorem'. (For a similar example, click here.)

For me this is so much better than getting your students to plough through exercise after exercise with little over-arching motivation. I would say that it is the difference between working out alone at the gym and taking part in some well-matched team game. (Although I concede that some people do like to pump iron at the gym!)

Overall, this worked well as a first-risp-of-the-year. The reality of the lesson was that students drifted in over a period of half an hour as they were gradually given their timetables. Using a risp is helpful in this situation: incoming students join a working atmosphere, while there is no danger of the early students running out of work. This risp required no introduction: everybody recognised that they already had most of the skills required to get going.

So how would a typical attempt at this look?
Suppose I choose the numbers 2, 3 and 4.

$$(x + 2)(3x + 4) = 3x^2 + 10x + 8$$
$$(x + 2)(4x + 3) = 4x^2 + 11x + 6$$
$$(x + 3)(2x + 4) = 2x^2 + 10x + 12$$
$$(x + 3)(4x + 2) = 4x^2 + 14x + 6$$
$$(x + 4)(3x + 2) = 3x^2 + 14x + 8$$
$$(x + 4)(2x + 3) = 2x^2 + 11x + 12$$

Adding the right hand sides gives $18x^2 + 70x + 52$, which is $2(9x + 26)(x + 1)$

It turns out that the right hand side will always factorise with $(x + 1)$ as a factor, a fact that students can discover by comparing notes with their partners.

There are a number of helpful points for the teacher here:
1. the exercise is self-checking. If a student ends up with a quadratic expression that does not factorise with $(x + 1)$ as a factor, they must have made an error, most likely in their expanding.
2. if a student wants help with factorising, the teacher is aided by the fact that one of the factors is fixed.
3. the teacher can use the Factor Theorem to detect a student mistake quickly. If presented with
$18x^2 + 70x + 48$, then $f(-1)$ is not zero, so there has been a mistake.

For the general proof, algebra is needed.

$$(x + a)(bx + c) = bx^2 + (ab + c)x + ac$$
$$(x + a)(cx + b) = cx^2 + (ac + b)x + ab$$
$$(x + b)(ax + c) = ax^2 + (ab + c)x + bc$$
$$(x + b)(cx + a) = cx^2 + (bc + a)x + ab$$
$$(x + c)(ax + b) = ax^2 + (ac + b)x + bc$$
$$(x + c)(bx + a) = bx^2 + (bc + a)x + ac$$

and so the total is $2(a+b+c)x^2 + 2(ab+bc+ca+a+b+c)x + 2(ab+ac+bc)$
which factorises to $2((a + b + c)x + (ab + ac + bc))(x + 1)$

As theorems go, simple fare. But I think my students experienced this risp as a kind of 'magic trick'. There was definite surprise amongst my class to find that everyone had $(x+1)$ as a factor. I could feel curiosity building, something that makes A Level maths come alive.

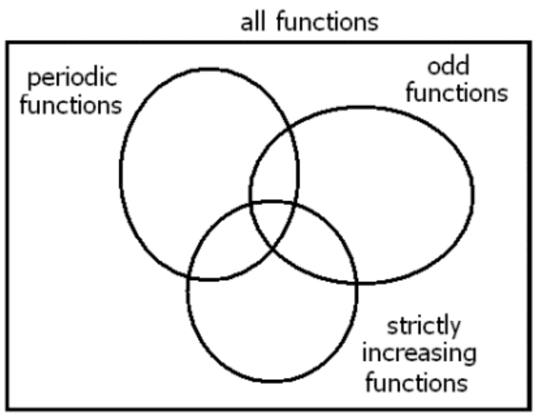
Risp 4: Periodic Functions

Write down a periodic function.
Then another. Then another...

Can you write down a periodic function with period 10?
And another? And another?...

What do we actually mean by the period of a function?

Can you place a function into each of the regions below?



Given a periodic function f(x), let us call its period 'per(f)'.

Pick two prime numbers p and q.
If per(f) = p, and per(g) = q, what is per(f + g)?

If you add two periodic functions, do you always get a periodic function?

Suppose that f and g are periodic functions.
Are the following statements always true, sometimes true or never true?

1. per(f x g) = per(f) x per(g)

2. per (kf) = k per(f)

3. per (f + g) = per (f) + per (g) [hard!]

Risp 4: Periodic Functions

Teacher notes

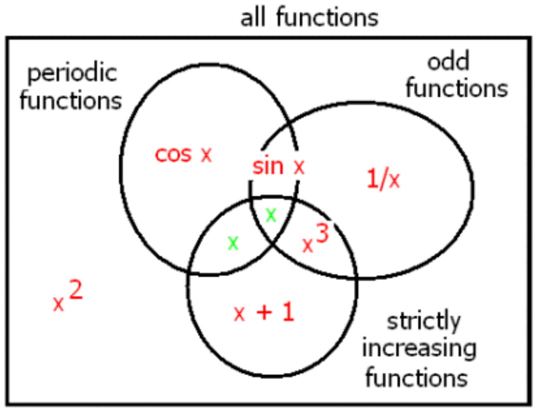
Suggested Use: to consolidate/revise functions (especially periodic)

Skills included:
Functions in general: periodic, odd, even functions
The transformation of graphs: deriving the graph of y = kf(x) from y = f(x)
the composition of functions

This activity includes several useful techniques for risp-construction. The "Give me an example of...and another...and another..." idea I first met at a workshop led by Anne Watson. We were asked to think of two numbers with a difference of 9, and another pair, and another pair... What effect would this have on you? Does the question become easier or more difficult if you change the number 9? I found the first example came came trivially enough, but each time I was asked for another case, I felt implicitly challenged to come up with something significantly new, maybe some way of looking at the problem that my colleagues would not have found. We then moved on to pairs of numbers differing by π. Comparing notes afterwards, we found my examples had been of the form n and n - π, while my partner had chosen examples of type nπ and (n-1) π. So there is more than one way to skin a cat...

There is much that arises naturally from this risp: Does it make sense to say per (non-periodic function) = 0? (I am grateful to Ian Short for asking: is a constant function a periodic function? What is per(f) for a constant function?) My students happily came up with sin x, cos x and tan x as periodic functions, but I like to introduce x - [x] too, where [x] is 'the integer part of x.' This 'saw-tooth' function has period 1. Converting these into functions with period 10 is clearly a vital skill, yet some found this hard. Graphics calculators are a huge help with this risp, allowing students to try ideas out.

The three-property, eight-region Venn diagram is another easy way to generate richness very simply. This will work wherever you have a set of mathematical objects with several different attributes they may or may not possess. With younger children you could pick the natural numbers up to 20, and the attributes 'prime', 'even' and 'square'. Or when dealing with sequences, you could have the attributes 'oscillating', 'periodic' and 'convergent'. For our current risp, possible answers are below:



Should I write per(f) or per(f(x))? There is a looseness of notation in my presentation of this risp, which might make some feel nervous. If there is a choice between being rigorous and meaningful, I generally go for the latter. Here I wanted to get across the idea that while a function is usually a function of something, there are times when a function can be the something!

If you add two periodic functions, do you always get a periodic function? The perhaps surprising answer is, "No." Consider sin x and sin (√2 x). The function sin x has period 360, while sin √2 x has period 360/ √ 2. So if their sum is periodic, 360 m = 360 n/ √ 2 for some natural numbers m and n. This gives √ 2 = n/m, which can lead to a highly profitable discussion!

Mark Cooker helpfully shared this problem, one that he uses with his undergraduate students: "Show that f(x) = cos(x) + cos(x √2) is not periodic." Mark's solution: one answer is to show that although f(0) = 2, f(x) is never again equal to 2.

Yet another risp-builder can be seen in the final section: "Are the following statements always true, sometimes true, or never true?" Suppose you are working in a topic area where there are certain classic boo-boos that every class will make at some stage. Why not put these errors centre-stage alongside some rather truer statements, and ask students to sift them into the above three categories?

For the risp we have here:

1. per (sin x cos x) = 180, not 360 times 360. However, per ((x - [x]) cos x) = 360 = 1 times 360. So 1. is sometimes true.

2. Generally, per (kf) = per (f). So as k cannot be 1, 2. is never true.

3. Given f with per(f) = a, and g with per(g) = b.

If per (f + g) = a + b, then a + b = ma, and a + b = nb,
for some natural numbers m and n. [This needs a little justification!]
So m = 1 + b/a, and thus b/a is a natural number.
But n = 1 + a/b, so a/b is a natural number also.
So a = b, but then per(f + g) = a, and not 2a. Contradiction!

Is this a bit of a surprise? per (sin x + cos x) = 360 < 720, yet per (sin 5x + cos 12x) = 360 > 72 + 30. Isn't there somewhere in the middle where equality holds?

A further extension could look at the composition of functions.

The questions above arise naturally, yet they do not have solutions that many A Level students can be expected to find. Does this make this risp too dispiriting to set? Are the harder questions included more for the pleasure of the teacher than to address the needs of the students? Could this be an example of 'risp lust' in action? (see "What can go wrong with a risp...") Every teacher should ask themselves these things before embarking on any risp.

Risp 5: Tangent through the Origin

(You will need a graphing package for the following.)

Pick a number, any number, between 0 and 5, & call it a.

Square a and add x^2. Draw the graph of y = this.

Now by experimenting, try to find $y = kx$ such that this line touches the graph.

How could we find where the line and the curve touch?

Are a and k related?

Try starting with other values of a: can you find a general rule?

Can you prove your conjecture?

Now pick any value for a, and any value for b.

Where does $y = kx$ touch $y = x^2 + bx + a^2$?

Risp 5: Tangent through the Origin

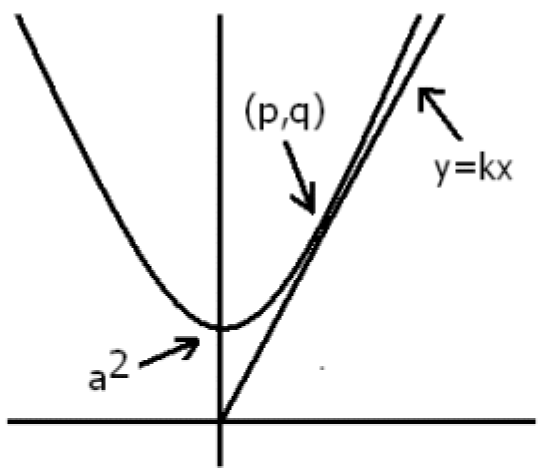
Teacher notes

Suggested Use: to revise coordinate geometry

Skills included:

co-ordinate geometry of straight lines and parabolas
simultaneous equations: solving a line and a curve
simple differentiation
equal roots for a quadratic equation
constructing a logical argument

One of the crucial things in setting a risp is to pick the right time in the year. Will your students have the maturity to tackle this now? How much prior knowledge do you want your students to have? For this risp, I chose a set of students nearly two terms through their AS year. They were comfortable with coordinate geometry, they had covered simple differentiation, and they could succeed in solving some quite hard sets of simultaneous equations. This risp turned into a chance to practice all of these skills, as well as those of sound logical reasoning.



The best thing to do is to draw the parabola, then use the Constant Controller to adjust k so that touching is achieved. Eventually both a and k can be entered as constants to be varied.

How to deal with this mathematically? We can write down the following equations:

(p, q) is on the curve, which gives $q^2 = p^2 + a^2$

(p, q) is on $y = kx$ which gives $q = kp$

The gradient of the curve at p is k, which gives $k = 2p$.

Thus $q = 2p^2$, and so $p^2 = a^2$, so $p = \pm a$.

Thus $q = 2a^2$, and $k = \pm 2a$.

The other way to see this is by saying $y = kx$ and $y = x^2 + a^2$ must give equal roots when solved together.

To demonstrate this method on the second problem:

solving $y = kx$ and $y = x^2 + bx + a^2$ together,

$kx = x^2 + bx + a^2$, so $x^2 + (b - k)x + a^2 = 0$

Equal roots mean $(b - k)^2 = 4a^2$, so $b - k = \pm 2a$, so $k = b \pm 2a$

Everyone in the group was able to spot the first relation ($k = \pm 2a$) by experiment, and most were able to find the second ($k = b \pm 2a$) with a little help. Justifying this was a good exercise.

When I have a bad day with a risp, when students do not warm to an activity as I had hoped, when they complain that I am asking too much of them, I sometimes doubt this whole project. When that happens, I think back to trying this particular task, and I remember why I am doing this. Sitting in pairs at computers running Autograph, my students became engrossed that day as the room gradually fell silent. There was an intensity as they worked, a blessed atmosphere in which they seemed to forget that I was in the room. For half an hour, the maths was everything. Then the chance came to put together what had been learnt and to build upon it. At the end of the lesson, one student stayed behind to try to express how good it had felt. So it wasn't just me...

Risp 6: The Gold and Silver Cuboid

Put the equation $y = 8x^3 - px^2 + qx - r$ into your graph plotter.

Now adjust p, q and r using the Constant Controller.

Can you find positive values for p, q and r so that the equation $y = 0$ has three positive solutions a, b and c?

Note down p, q and r, and your values for a, b and c to 4sf.

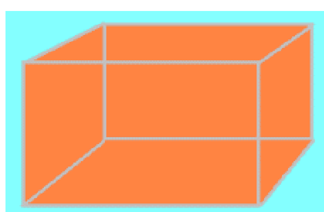
Sketch the cuboid with sides 2a, 2b, and 2c.

Find V (= volume), S (= surface area) and E (= edge-length)

for this cuboid.

Can you find a connection between p, q and r and V, S and E?

Can you prove this will always work?



A cuboid uses exactly 10 cm of silver edging along all of its edges and exactly 3 cm² of gold paint to cover its surface area.

What is the maximum V can be, and what are the sides of the cuboid in this case? (to 3sf)

What is the minimum V can be, and what are the sides of the cuboid in this case? (to 3sf)

Risp 6: The Gold and Silver Cuboid

<u>Teacher notes</u>

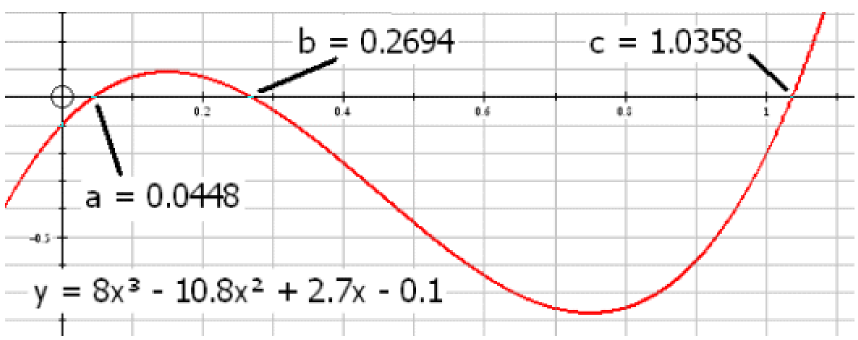
Suggested Use: to revise ideas of polynomials/curve-sketching

Syllabus areas covered:

Basic algebra, including expanding brackets
Knowing how to solve an equation graphically
Knowing how to sketch the graphs of polynomial functions, especially in factorised form
Knowing how to sketch the curve of y = f(x) + a given the curve y = f(x)

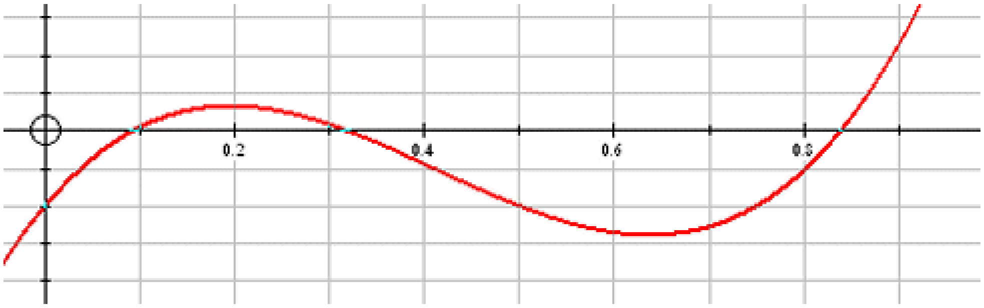
I have changed this risp a great deal since I first added it to the site. Each time I tried the original with students, the logic of the activity felt uneasy. The task was too linear: there was not enough preliminary 'messing about'. I was being too prescriptive - "I have something in mind that I wish you to discover, and by golly, you will discover it."

This revised approach is much more rispy I think. It starts with students playing on a computer: the teacher is not the centre of attention, and the initial task is to get the screen to look a certain way, which is accessible to everyone.

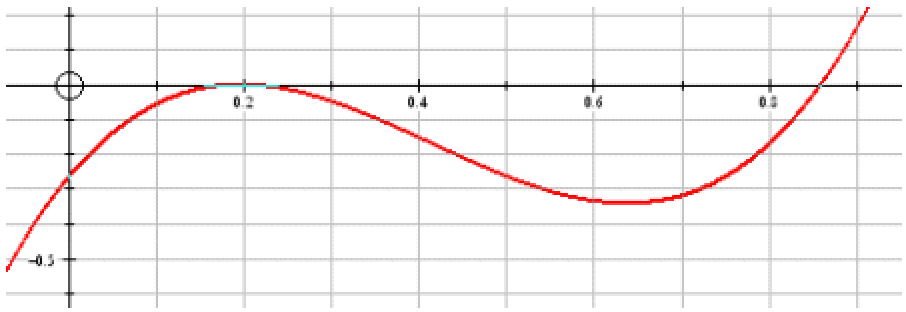


Here we have an example of p, q, r, a, b, and c all being positive. So the cuboid has sides of length 0.0896 cm, 0.5388 cm, and 2.0716 cm, which gives V = 0.1 cm^3 (= r), S = 2.7 cm^2 (= q), and E = 10.8 cm (= p). Surely a mathemagical trick to impress any student, however sceptical. To prove this always works, you might invite a student to expand (2x - i)(2x - j)(2x - k). *(Note: (2x - i)(2x - j)(2x - k) = 0 has roots i/2, j/2 and k/2.)* This gives $8x^3 - 4(i + j + k)x^2 + 2(ij + jk + ki)x - ijk$, or $8x^3 - Ex^2 + Sx - V$, where E, S and V are the edgelength, surface area and volume of the cuboid with sides i, j and k.

So now we can turn the problem around. If E is 10 cm, and S is 3 cm^2, we have the graph $y = 8x^3 - 10x^2 + 3x - V$. Suppose we choose V = 0.2 cm^3 to start with:



To maximise V, we need to increase the value 0.2, which will lower the curve, until we only just have three real solutions, that is, until the curve touches the x-axis.

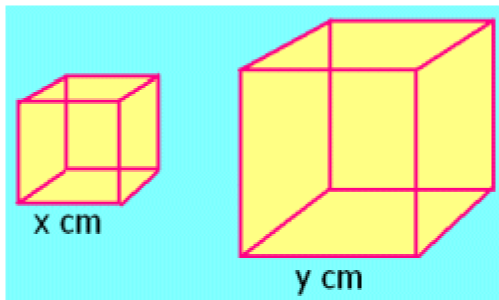


This gives a maximum value for V of 0.26408 cm^3, when the sides are 0.392 cm (twice) and 1.716 cm. V takes a minimum value of 0 cm^3 when the sides are 1 cm, 1.5 cm and 0 cm.

If you have 3-D Autograph available, you can draw $4(x + y + z) = 10$ and $2(xy + yz + zx) = 3$, and then add $xyz = k$. When is k a maximum if we still have an intersection point?

Risp 7: The Two Special Cubes

**You are presented with two cubes,
one of side x cm, the other of side y cm, with $x < y$.**



**V is the total volume of the two cubes.
S is the total surface area of the two cubes.
E is the total edge-length of the two cubes.**

You are given that E, S and V (taken in some order) are in arithmetic progression.

**What is the maximum possible value for y?
If y takes this value, then what is x?**

[Hint: a, b, and c are in arithmetic progression iff $a + c = 2b$.]

Risp 7: The Two Special Cubes

Teacher notes

Suggested Use: to introduce the idea of implicit differentiation

The volume, surface area and total edge-length of a cube and the ways in which they can be connected are something of a leitmotif in my risps. They seem a natural thing to calculate when faced with a solid, and the way you come across something linear, quadratic and then cubic in turn is often helpful. Even the following beautifully simple question is a rich one:

"There are six ways to write V, S and E in order of size.
Can you find a cube for each order?
What about a cuboid?"

(With thanks to Rachel Bolton)

So how did I start on this risp? I began by asking students to think a bit about the problem. I watched some useful work: some students used the hint to arrive at the fact that:

$$E = 12x + 12y,\ S = 6x^2 + 6y^2, \text{ and } V = x^3 + y^3.$$

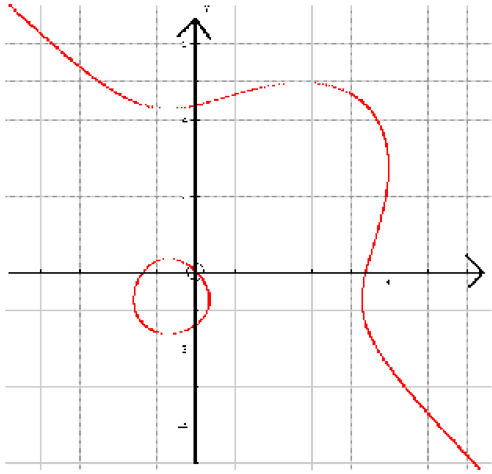
So one of:

$$12x + 12y + 6x^2 + 6y^2 = 2x^3 + 2y^3$$
$$6x^2 + 6y^2 + x^3 + y^3 = 24x + 24y$$
$$12x + 12y + x^3 + y^3 = 12x^2 + 12y^2$$

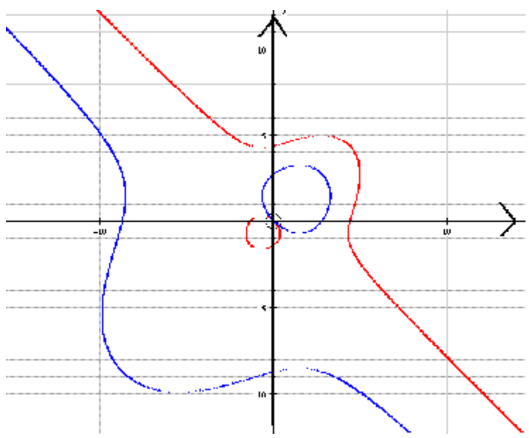
must be true.

Beyond this, what could they do? Time to introduce a graphing package, one with a facility for drawing implicitly-defined functions.

What would $12x + 12y + 6x^2 + 6y^2 = 2x^3 + 2y^3$ look like? I would defy anyone to sketch this without computer help, although it would be worth noting that we do expect a graph that is symmetrical in $y = x$.

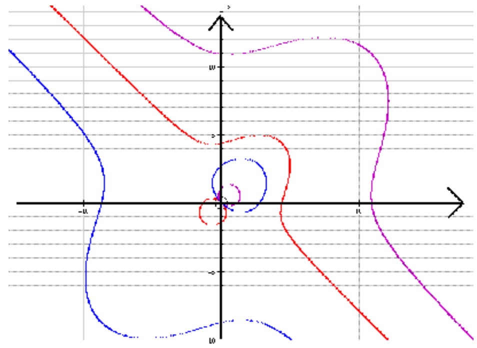


My students reacted with delight to this, as something unexpected and offbeat. So which area of the diagram are we interested in? My students quickly saw that the first quadrant is the answer to this - x and y must be positive. So we do have a maximum to find here. But might the other equations yield something larger? Adding the graph of $6x^2 + 6y^2 + x^3 + y^3 = 24x + 24y$ gives the following:



So another maximum in the first quadrant here, but not a larger one. We can now add the third curve, $12x + 12y + x^3 + y^3 = 12x^2 + 12y^2$ (*)

So this shows us the value of x we seek. We can home in on the stationary point, but the rather grainy drawing of this implicit graph limits the accuracy we can obtain (fortunately, given that we are heading towards finding a really accurate solution by differentiating.) By inspection we arrive at 12.1... for y and 7.46... for x.



So, how would we differentiate to find the maximum point in this case? We can see the point we want to find, but we have never tried to find y' for something like this, where we cannot make y the subject. A perfect opportunity has arisen to introduce implicit differentiation. If we carry out the procedure here we arrive at:

$$y' = (8x - x^2 - 4)/(4 + y^2 - 8y)$$

So $y' = 0$ if $x^2 - 8x + 4 = 0$, or $x = 7.464...$, or $0.5358...$

We can see the two maxima for these x-values on the graph. So the x we seek is $4 + \sqrt{12} = 7.464...$, and to find the corresponding y-value, we substitute back into * getting:

$$y^3 = 12y^2 - 12y + 163.138...$$

Happily, the obvious rearrangement yields us the root after iteration:

$$y = \sqrt[3]{(12y^2 - 12y + 163.138...)} \text{ tends to } y = 12.12043946....$$

So we end up with a value for x of 7.46410 (6sf) and for y of 12.1204 (6sf).

I would mention the word 'scaffolding' at this point. It is a term that was introduced into educational literature by Wood, Bruner and Ross in 1976. The connotations are of support while building is in progress, but with the obvious rider that the support is intended to be temporary. My Assistant Principal described 'scaffolding' as "an awful term" on a recent Inset day, but I am not so sure. In this particular risp, I found that a lot of scaffolding was necessary at the start, but that it could be swiftly dismantled.

This risp uses the fact that motivated theory is remembered theory. The problem carried my group along to the point where the idea that we would need the maximum of an implicitly defined function seemed quite natural. They were then able to approach more prosaic questions with equanimity.

Risp 8: Arithmetic Simultaneous Equations

...1, 3, 5, 7, 9, 11....
...-16, -5, 6, 17, 28, 39...
...78, 76, 74, 72, 70, 68...

Each of the above sequences is called ARITHMETIC, because the terms go up or down by a constant amount each time.

Pick six consecutive terms from an arithmetic sequence, and place them in order into the squares below. (Keep the numbers as simple as you can to start with!)

$$\square x + \square y = \square$$
$$\square x + \square y = \square$$

Now solve the pair of simultaneous equations you have created.

Do the same thing twice more. (You could pick an arithmetic sequence containing negative numbers this time.)

What do you discover? Can you make a conjecture? Can you prove it?

What happens if you take the first three numbers from one arithmetic sequence, and the next three from another?

Do you want to revise your conjecture?

What if our starting sequence is geometric? Fibonnacci? Experiment...

Risp 8: Arithmetic Simultaneous Equations

Teacher notes

Suggested Use: to consolidate/revise Simultaneous Equations

Skills included:
proof, mathematical language and argument

Risps can be a tremendous help when you start a topic where your students possess a wide range of prior learning. Typically an AS class will contain those who have C on the Intermediate paper as well as an A* on the Higher. Somehow you need to find a way to cover simultaneous equations, which is a vital skill on the AS syllabus as well as on both the Higher and Intermediate ones. Your approach must enable you to support the weaker students whilst still stretching the stronger ones. You could have three strands of material, easy, medium and hard, and let students choose. But maybe you don't have time to create three strands of material. Some teachers might throw up their hands and say, "I am being asked to square the circle here." Some would say that students who have only a C should not start AS Mathematics.

As I write, the Standards Unit have just brought out a carefully piloted collection of ideas for A Level mathematics teaching that try to address exactly this kind of problem. The emphasis is on active learning, on discussion, on students sharing their knowledge with each other in non-threatening situations. My early attempts to try the SU lessons plans have been highly productive. Weaker students can no longer sit nodding passively in a corner whilst understanding little. Instead, they are encouraged to spread their wings, to be original, to debate, to make mistakes, in short, to express themselves. Sadly, this may be a new experience for them in a mathematics classroom. Within this fresh atmosphere, all students seem to find a confidence that feeds back into every other way of dealing with their mathematics.

One SU technique is poster work. Setting a group the task of preparing an instructive poster on simultaneous equations that they are then prepared to present to the class can change around tedious revision of a topic into a discussion where everybody wins. The weaker students learn in engaging non-linear ways, while the stronger students find that the best way to be sure you have mastered a topic is to teach it to your peers.

This risp can work in a similar kind of a way. Suppose a student chooses the simplest possible set of six numbers from an arithmetic sequence, 1, 2, 3, 4, 5 and 6. These generate:

$$x + 2y = 3$$
$$4x + 5y = 6$$

Note that the student has created this example for themselves, which adds a real motivation to solve it. The solution is $x = -1$, $y = 2$. Teacher time at this point should be devoted to the weaker students: can you solve these equations? What happens if we interpret these as graphs? What if we make these equations even simpler? Taking -2, -1, 0, 1, 2 and 3 as your terms generates:

$$-2x - y = 0$$
$$x + 2y = 3$$

Once again we get the solution $x = -1$, $y = 2$. The truth begins to dawn that building a pair of simultaneous equations with coefficients from an arithmetic sequence in this way always gives you (-1,2) as a solution. This has a helpful side for the teacher. Often the problem with improvised simultaneous equations is that they have nasty solutions. Students get bogged down with fractions, and the techniques they should be practising get lost. Here you can have huge numbers, and stretch equation-solving powers to the limit, safe in the knowledge that the answer will be a clean (-1, 2) every time.

Meanwhile the stronger students will have been turning to algebra.

$$ax + (a + d)y = a + 2d$$
$$(a + 3d)x + (a + 4d)y = a + 5d$$

Subtracting, we get $3dx + 3dy = 3d$, so $x + y = 1$. So $y = 1 - x$, and substituting back into the first equation gives $x = -1$, $y = 2$. (This really is an activity where everything simplifies nicely.)

So we have that choosing coefficients in this way gives a solution of (-1, 2). Is the reverse implication true? No, since

$$4x + 5y = 6$$
$$8x + 7y = 6$$

have the solution (-1, 2) too. A little revision gives the mini-theorem:

The first three coefficients come from an arithmetic sequence, and so do the second three, if and only if the solution to the two equations is (-1, 2).

There are all sorts of extensions to this work. Fibonacci sequences give the answer (1,1), for fairly clear reasons. Why not look at simultaneous equations in three variables? If students want to go further into this, they will appreciate the use of a graphics calculator to do the donkey work for them. (In the work above the point is to practice simultaneous equation techniques, so that should not be allowed!)

I am grateful to Alan Fry for suggesting the following:

Note that

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

has the solution

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

so this risp lends itself to good determinant practice!

$$x = \frac{\begin{vmatrix} (m+2k) & (m+k) \\ (m+5k) & (m+4k) \end{vmatrix}}{\begin{vmatrix} m & (m+k) \\ (m+3k) & (m+4k) \end{vmatrix}} = \frac{\begin{vmatrix} k & (m+k) \\ k & (m+4k) \end{vmatrix}}{\begin{vmatrix} m & k \\ (m+3k) & k \end{vmatrix}} = \frac{\begin{vmatrix} k & (m+k) \\ 0 & 3k \end{vmatrix}}{\begin{vmatrix} m & k \\ 3k & 0 \end{vmatrix}} = -1$$

$$y = \frac{\begin{vmatrix} m & (m+2k) \\ (m+3k) & (m+5k) \end{vmatrix}}{\begin{vmatrix} m & (m+k) \\ (m+3k) & (m+4k) \end{vmatrix}} = \frac{\begin{vmatrix} m & 2k \\ m+3k & 2k \end{vmatrix}}{\begin{vmatrix} m & k \\ m+3k & k \end{vmatrix}} = \frac{\begin{vmatrix} m & 2k \\ 3k & 0 \end{vmatrix}}{\begin{vmatrix} m & k \\ 3k & 0 \end{vmatrix}} = 2$$

Risp 9: A Circle Property

I'm going to pick a number, let's say...12.

Now you pick two numbers, any numbers, let's call them p and q, that multiply to 12.

(Make sure everyone in your group picks different numbers.)

Now pick two numbers, any numbers, call them s and t, that multiply to -12. (Again, don't all pick the same numbers.)

Now plot the point A = (p, s) and the point B = (q, t), and using compasses, draw the circle that has AB as diameter. [Careful! A and B are not (p,q) and (s,t)!]

Do you notice anything unusual about your circle? (If you don't, compare your circle with those your colleagues have drawn. What do they all have in common?)

Can you find the equation of your circle? Does this confirm your findings?

What happens if we change the starting number? Can you make any conjectures? Can you prove them?

Risp 9: A Circle Property

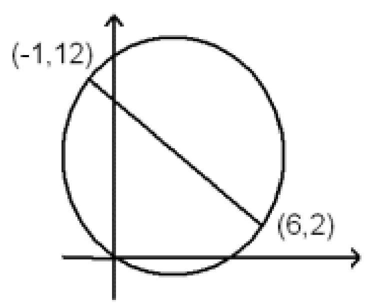
Teacher notes

Suggested Use: To consolidate/revise ideas of circles in coordinate geometry

Skills included: Co-ordinate geometry: the equation of a circle
the midpoint of two points the distance between two points

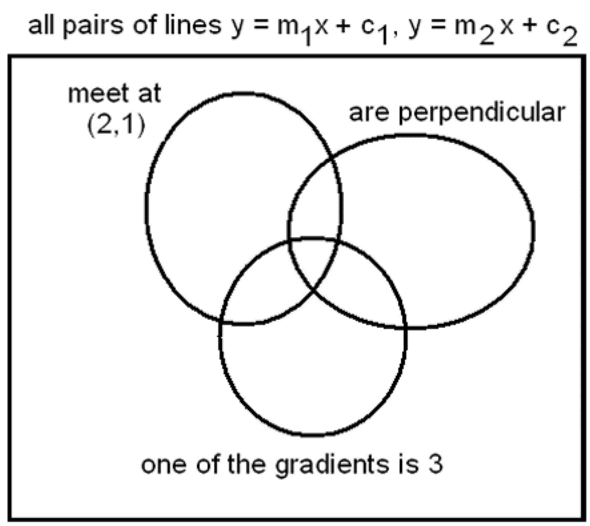
This activity is a good example of how risp-creation might take place. Suppose you have covered circles in co-ordinate geometry, and you would like to consolidate the work. You look for a question: how about, what is the equation of the circle with A = (c, d) and B = (e, f) as end points on a diameter? To answer this, we need to find the midpoint of AB (a handy bit of revision), and then the distance from here to either A or B to get the radius (again a vital skill). Now we can construct the equation of the circle (more good practice), which finally simplifies to:
$x^2 + y^2 - (c + e)x - (d + f)y + ce + df = 0$.

So now it might occur to us that if we choose df = -ce, the circle will go through the origin. Can we use this fact at the start of the activity, rather in the manner of a magician picking numbers? Now we have a risp.



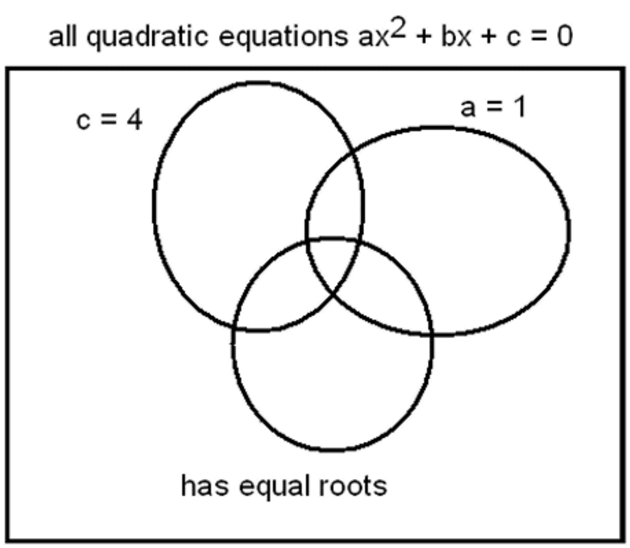
By turning the question around ('rispifying' it), we ask our students to practice the same skills that we did in its construction, but this time with something to discover. When I tried this with my students, they enjoyed the sense of purpose the activity has. However, they were slow to spot what made their circles special, which led to my advice about comparing with others. They also became confused with my notation: initially I used (a, b) and (c, d) as diameter end-points, which became mixed up with the centre (a, b) in the standard formula for the equation of a circle. (It is a good idea not to use (p, q) and (r, s) for similar reasons.) The stronger students did begin to experiment with algebra and the general case. When I presented this to the group, I felt it would be too much to go through this with everyone, although the work does simplify nicely. On another day, I might have risked it...

Risp 10: More Venn Diagrams



Can you find a pair of lines for each of the eight regions above?

* * * * * * * * * * * * * * *



Can you find a quadratic equation for each of the eight regions above?

Risp 10: More Venn Diagrams

Teacher notes

Suggested Use: To consolidate/revise ideas about anything!

Skills included: lines in co-ordinate geometry
quadratic equations
anything you like

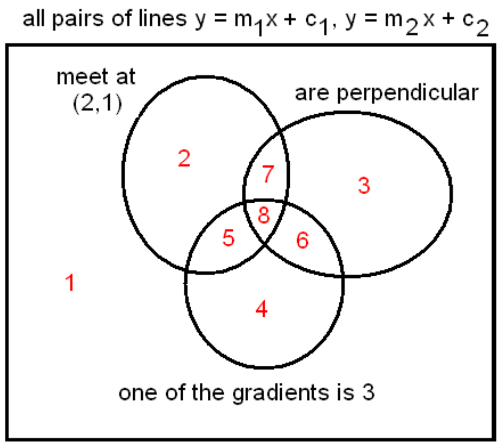
The usefulness of Venn Diagrams as a risp has already been mentioned (see Risp 4: Periodic Functions). I revisit the subject again because their penetrating nature has been impressing itself upon me more and more recently. To be able to fill in an eight-region diagram shows a really deep understanding of the distinctions that a good student has to be able to make, while the weaker student finds a picture such as this much less threatening than a page of exercises.

"So you can't think of a way to start?" (On the quadratic equation diagram, let's say.) "Just write down any quadratic equation, any at all." (Even the weakest student should be able to manage this.) "Now, where do you think it goes?" Once again, the fact that the student has thought up their own equation gives them a motivation to place it properly. And it must go somewhere! They must have written down a correct answer, if only they can correctly divine the nature of its correctness.

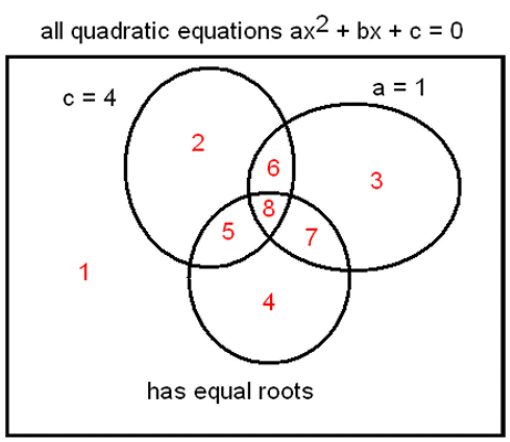
I reflect too on how easy this risp is for the teacher. Draw a box and three circles on the board, with three properties and an overall description for a set of objects, and you are away. No photocopying, no instructions to decipher, no cutting up, no projector to set up, and no network that may crash. Which is not to be a luddite: I love the laptop/projector arrangement and the possibilities it opens up, but my feeling is that risps that do not sympathise with the realities of a teacher's existence will never get used. I sometimes trial new materials for others, and my heart sinks when I see another slab of documentation flopping through the letterbox. Materials for the classroom should be completely intuitive, virtually running themselves. Those who write these materials have often left the classroom more or less gratefully, and their memories seem to be remarkably short over what it actually felt like.

There is sometimes too an annoyance on the part of these writers if you do not deliver the trial lesson in exactly the way that they imagined it taught. With my risps, the reverse is true. I wave goodbye to my risps when I post them as if they are children leaving home. I offer no party line that I ask you to toe. I hope you will improve on my ideas, and I look forward to enjoying such improvements.

As regards this risp, possible solutions are here:



1. $y = x$, $y = 2x$
2. $y = x - 1$, $y = 2x - 3$
3. $y = x$, $y = -x$
4. $y = 3x$, $y = x$
5. $y = 3x - 5$, $y = x - 1$
6. $y = x - 1$, $y = -x + 3$
7. $y = 3x$, $y = x/3$
8. $y = 3x - 5$, $y = -x/3 + 5/3$



1. $2x^2 + 3x - 1 = 0$
2. $2x^2 + 3x + 4 = 0$
3. $x^2 + 2x + 5 = 0$
4. $(2x-1)^2 = 0$
5. $(3x-2)^2 = 0$

6. $x^2 + 2x + 4 = 0$
7. $(x-3)^2 = 0$
8. $(x-2)^2 = 0$

Ideas for other eight-region Venn diagrams?

Overall description: all circles $(x - a)^2 + (y - b)^2 = r^2$
Property 1: circle goes through (3, 0)
Property 2: circle goes through (-3, 0)
Property 3: $b = 4$

Overall description: all quadratic expressions $ax^2 + bx + c$
Property 1: remainder 1 when divided by $x + 1$
Property 2: $x - 1$ is a factor
Property 3: $x - 2$ is a factor

Overall description: all binary operations on the set of real numbers excluding 0
Property 1: commutative Property 2: associative Property 3: closed

And the list goes on...

Risp 11: Remainders

**Pick any two integers, call them a and b,
so that $a - b > 12$.**

**Now pick any polynomial expression f(x) with integer coefficients.
(A quadratic or cubic to start with.)**

**What is the remainder when you divide f(x) by x - a?
[Call this R_a.]**

**What is the remainder when you divide f(x) by x - b?
[Call this R_b.]**

And what is the remainder when you divide $R_a - R_b$ by a - b?

Can you make a conjecture here?

Can you test this?

Can you prove it?

Risp 11: Remainders

Teacher notes

Suggested Use: To consolidate the Remainder/Factor Theorems

Sometimes less is more with risps. The activity above is really simple, yet it proved effective with my AS students. This starter once again provides good practice, here in the use of the Remainder Theorem, with a rough end point in view to provide a motivation. The final result might be scorned as ordinary, yet the proof of the Remainder Theorem itself can look fairly trivial at times, and it remains surprisingly useful.

Suppose a student chooses 31 and 19 as their integers. (The exercise will work with numbers that are close together, but if they are too close it is harder to work out what is going on.) They then pick $x^3 + x - 1$ as their polynomial.

Dividing by $(x - 31)$ gives a remainder of $31^3 + 31 - 1$, or 29821. Students can try working with some pretty big numbers here, which increases the drama.

Dividing by $(x - 19)$ gives a remainder of $19^3 + 19 -1$ or 6877. So $R_a - R_b = 22944$.

$$a - b = 12, \text{ and } 22944/12 = 1912 \text{ exactly.}$$

So do we always get the remainder zero? Students soon compare notes on this, and wonder how to show the natural conjecture is true. It is easiest here to start with the general quadratic as the chosen polynomial.

So choosing $px^2 + qx + r = f(x)$, the remainder on dividing $f(x)$ by $(x - a)$ is $pa^2 + qa + r$, and on dividing $f(x)$ by $(x - b)$ is $pb^2 + qb + r$.

So $R_a - R_b = p(a^2 - b^2) + q(a - b) = (a - b)(pa + pb + q)$.

So $(a - b)$ is a factor of $R_a - R_b$.

Is $(a - b)$ a factor of $a^3 - b^3$? Of $a^4 - b^4$? Students can be encouraged to pattern-spot here.

To generalise to all polynomials, we need to show that $(a - b)$ is a factor of $a^n - b^n$. One of the pleasing things about this risp is that this is simply a consequence of the Factor Theorem. $a - b$ is zero when $a = b$, and putting $a = b$ into $a^n - b^n$ gives zero, so $a - b$ must be a factor of $a^n - b^n$. There is a sense of having come full circle. Factorising $a^n - b^n$ fully is also a helpful exercise for any A Level student.

Risp 12: Two Repeats

In an idle moment,
Luke picked two numbers x and y with $0 < x < y < 1$.
He wondered how to combine these simply,
so with the help of a calculator,
he wrote down the numbers:

$x + y$, xy, x/y, y/x, $x - y$ and $y - x$.

He realised to his surprise
that he had written the same number down twice.

To his further surprise,
he noticed he had written a second number down twice.

What were his starting numbers?

Risp 12: Two Repeats

Teacher notes

Suggested Use: to consolidate/revise algebra

Syllabus areas covered:
Changing the subject of a formula
Graphical solution of equations
Simultaneous equations
Solving a quadratic
Manipulating surds

Sometimes there is a lesson that is a bit different for some reason. Maybe half the class have been entered for an exam, or the snow has drifted that day. I find it is always a good idea to have activities up your sleeve to cover for this. Activities like this risp, a piece of mathematical circuit training that consolidates useful skills without requiring any extra theory that your absentees will be missing. By the end of this task, your students will know they have been given a thorough algebraic workout.

So how might we start here? Students sometimes lose confidence in the face of a novel question such as this. How many possible pairs are there? It looks as if there might be $^6C_2 = 15$ possible pairs, leading on to $(^6C_2 \times {}^4C_2)/2 = 45$ pairs of pairs. The idea of checking out 45 possibilites makes the keenest mathematician's heart sink. But of course, we can start to rule out lots of these quite quickly.

Immediately we can see that $x - y$ is the only one of the six numbers that is negative, so it cannot be a repeat.

Can $x + y = y - x$? This gives $x = 0$, which is not allowed.

Can $xy = x/y$? This implies $y = \pm 1$, which again is a contradiction.
Similarly for $xy = y/x$.
Can $y/x = x/y$? Again, the contradiction is clear.

So the five remaining numbers fall into two groups,
$\{x + y, y - x\}$ and $\{xy, x/y, y/x\}$.
The first repeat must be $x + y$ = one of the second set,
and the second repeat must be $y - x$ = another one of the second set.

What if $x + y = xy$? $x + y$ is bigger than x, while xy is smaller than x. Yet another contradiction.

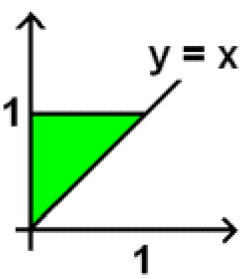
And if y - x = y/x, we have the problem that y - x is less than 1 while y/x > 1.

So we end up with just four pairs that might be true:

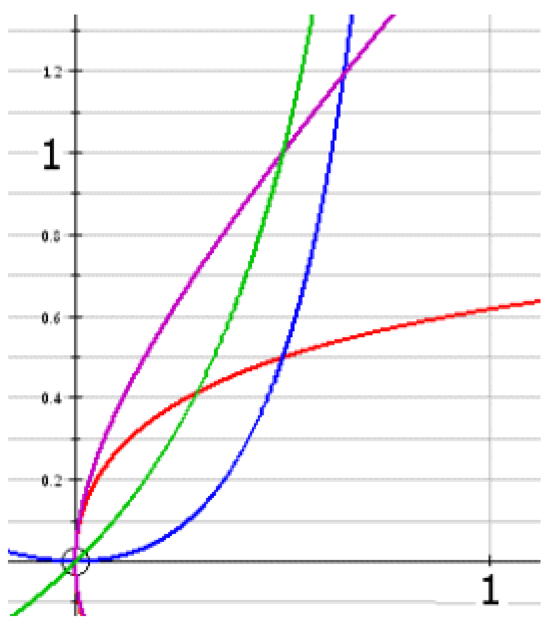
1. $x + y = x/y$, 2. $x + y = y/x$, 3. $y - x = xy$, and 4. $y - x = x/y$.

A much less daunting number than the 15 we started out with.

Where to now? Why not bring in our graphing package to draw these four curves? (Asking students to first change the subject to either x or y in each case provides a chance for some more excellent practice!) What kind of crossing points are we looking for? We must have two of the curves intersecting in the following region:



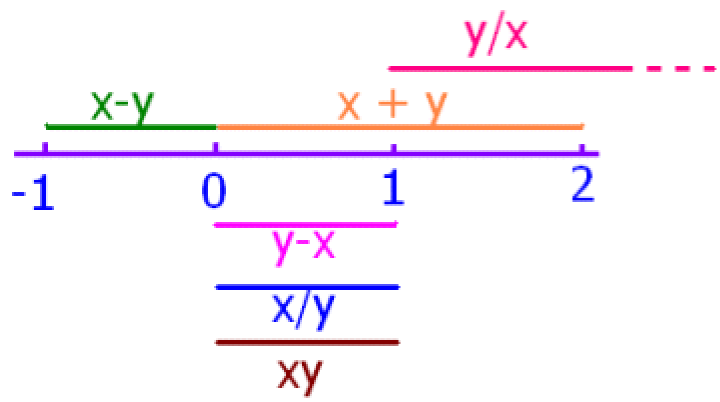
This is the picture your package will draw:



It is easy to read off the only two curves that meet in the region we are looking for:
$x + y = y/x$ (or $y = x^2/(1 - x)$) and $y - x = xy$ (or $y = x/(1 - x)$). Solving these together as simultaneous equations eventually gives $2x^2 - 4x + 1 = 0$. Thus $x = 1 - (1/2)\sqrt{2}$, and $y = \sqrt{2} - 1$.

If your students are not completely algebra-ed out by this stage, they can put these answers back into x + y, xy, x/y, y/x, x - y and y - x, and check that they do indeed give two repeats. This is excellent surd practice.

Another approach is to map out on a single diagram what ranges the six initial numbers might fall into.



We end up with four possible combinations to explore:

1. x + y = y/x, y - x = xy
2. x + y = y/x, y - x = x/y
3. x + y = x/y, y - x = y/x
4. x + y = x/y, y - x = xy

Multiplying the two equations in 1 gives $y^2-x^2 = y^2$ (contradiction).
Doing the same for 2 or 3 gives $y^2-x^2 = 1$, or $y^2>1$ (contradiction).
Whihc leaves 4 as the only viable option, yielding our unique numbers as above.

Overall, a problem with no excessively demanding theory that covers a remarkably wide range of algebraic techniques. It can be seen as the little brother of some of the advanced and nasty problems that go "Person A says, "I do not know the numbers are," to which B replies, "Now I know what the numbers are!" " Your absentees will hopefully arrive back to find their colleagues more confident and skilled in the basics.

Risp 13: Introducing e

(You will need a graphing package for this risp.)

What do the following situations have in common?

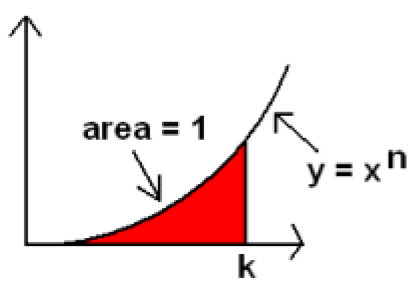
1. Draw $y = x^{1/x}$.

2. Draw $y = a^x$ together with its gradient function.

3. Draw $y = (1 + 1/x)^x$.

*** * * * ***

Pick a number between 0 and 3, call it n. Draw the curve $y = x^n$. k is the number so that the area between your curve and the x-axis between 0 and k is exactly 1.



Can you find k for your chosen value of n? Try to find k to three decimal places.

Now vary n within the interval [0, 3]. As n varies, k varies.
Can you draw up a table for n and k?
What would a graph of k against n look like?
What happens as n tends to infinity?
Would there be any stationary points?
What range of values can k take?
Can you use integration theory that you already know to explore this?

Risp 13: Introducing e

Teacher notes

Suggested Use: To consolidate/revise the integration of x^n

Skills included:
definite integrals
numerical integration including Trapezium Rule

This risp assumes that students have met the rule for finding $\int x^n dx$ for all n not equal to -1, and the idea that the area under a curve can be written as a definite integral.

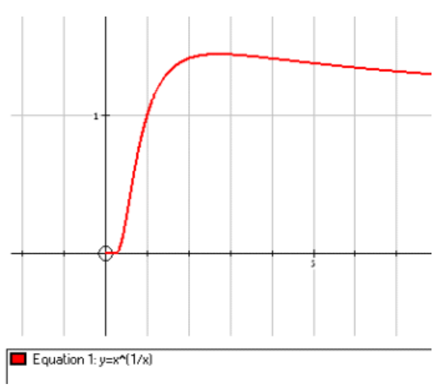
A table of values for n with their corresponding values of k can be constructed. Drawing up this table can take a lot of time, unless you use the Constant Controller to find k. [The following instructions apply to Autograph.] Put two points on the curve, one at x = 0 and one at x = k, by using the right-click option with only the curve selected. Select the two points, then use the *Find Area* option which you will find on right-clicking. Now adjust k as required. The final table might look like this:

n	0	0.5	1	1.5	2	2.5	3
k	1	1.310	1.414	1.443	1.442	1.430	1.414

A possible range for k for n in [0, 3] would seem to be (1, 1.444...). The graph of k against n seems to have a maximum. What value of n gives that maximum?

Turning to theory, if $\int_0^k x^n dx = 1$, then $[x^{(n+1)}/(n+1)]_0^k = 1$, so $k^{(n+1)}/(n+1) = 1$.
Thus $k^{(n+1)} = n+1$, and $k = {}^{(n+1)}\sqrt{(n+1)}$.

We can draw the graph of $y = {}^{x}\sqrt{x}$ using our graphing package.



The curve has a maximum when $x = e$, although the calculus for this would be hard for students setting out at AS Level. It is, however, a nice intuitive way to introduce the number e to students who have never come across it before, which hopefully the introduction accomplishes.

What if n ranges across $[0, \infty]$?
It would seem that k can be anywhere in the interval $(1, \sqrt[(e-1)]{(e-1)})$,
with the maximum occurring when $n = e - 1$.

Risp 14: Geoarithmetic Sequences

Pick two numbers between 0 and 1, call them a and b.

Generate a sequence from these numbers as follows:

1. Take a as the first term.

2. Add b to this to get the second term.

3. Multiply this by b to get the third term.

4. Add b to this get the fourth term.

5. Multiply this by b to get the fifth term.

And so on...

What are the first six terms of your sequence?

**What do you think will happen to the nth term as n gets larger?
What if we change a? Change b?**

***(You will find a brief program on a graphics calculator
useful to explore these questions.)***

**What happens if we remove the restriction
that a and b must be between 0 and 1?**

**What different types of behaviour
can a sequence like this exhibit?**

Can you predict what will happen for any given a and b?

Risp 14: Geoarithmetic Sequences

Teacher notes

Suggested Use: To introduce/consolidate/revise types of sequence

Skills included:
Types of Sequence, convergent, divergent, oscillating, bounded, periodic

Risp 2 tackles sequences, and here is another activity that does the same. The former is a superior consolidator, while this risp tends to work better as a starter on the subject. Your syllabus deserves a careful reading over what exactly terms like 'divergent' mean. I take 1, -2, 3, -4 , 5, -6... as being divergent and oscillating, and 3, 1, 4, 1, 5, 9... as being bounded, and oscillating, but not divergent. (How would you use the word 'chaotic' to describe a sequence?)

The syllabus we use [MEI] implicitly expects students to recognise six types of sequence:

1. convergent-monotonic
2. convergent-oscillating
3. divergent-monotonic
4. divergent-oscillating
5. periodic
6. bounded, non-periodic and non-convergent

How to introduce these? The good news is that this risp gives rise quite naturally to five from this list. My students started by having a play, as with all good risps.

For example, if we choose $a = 0.1$, and $b = 0.4$, then the sequence goes:

0.1, 0.5, 0.2, 0.6, 0.24, 0.64, 0.256, ...

As $n \longrightarrow \infty$, the terms get closer and closer to 2/3, 4/15.

Assuming we have 'convergence' here, we are effectively solving $(x + b)\, b = x$, so the sequence gets closer and closer to oscillating between $b^2/(1 - b)$ and $b/(1 - b)$. Notice that the starting value of a is irrelevant to what finally happens in this situation.

After a couple of experiments, students will tire of the calculation aspect of this activity, and the time comes for a calculator program (or Excel) to remove any drudgery.

I have no wish to teach anyone how to suck eggs, but below is a short program that will work on my graphics calculator to help this exercise along.

```
? → A : ? → B : Lbl 1 : A + B → C : C #
BC → A : A #
Goto 1 :
```

(# stands for the 'Print' command.)

So what happens as we choose any real value for b?

$1 \le b$, $u_n \longrightarrow \infty$ (divergent-monotonic)

$0 < b < 1$, $u_n \longrightarrow$ two 'limits', both positive (bounded/non-periodic/non-convergent)

$0 = b$, $u_n \longrightarrow 0$ (convergent)

$-1 < b < 0$, $u_n \longrightarrow$ two 'limits', one positive, one negative (bounded/non-periodic/non-convergent)

$-1 = b$, for example, 2, 1, -1, -2, 2, 1, -1, -2, 2, 1, ...
(the sequence is periodic, period 4) Note: the value of a does matter here!

$b < -1$, $u_n \longrightarrow +/- \infty$ (divergent-oscillating)

So the only type of sequence we cannot find through this work is the convergent-oscillating kind. A good extension question might be, "How could you tweak the sequence-generation rules here to give us our missing sequence?" Note that for $-1 < b < 0$ and $0 < b < 1$, we have two monotonic-convergent sequences intertwined, so to speak. Can students come up with ways of defining the nth term in terms of previous terms for each of these sequences?

Another nice question: given k, how many values of a and b give k as one of the limiting terms? If we take k = 2, then (a, b) pairs that work are:

(any, $-1 + \sqrt{3}$), (any, 2/3), (2, -1), (3, -1), (-1, -1), (-2, -1)

Risp 15: Circles Or Not?

Type $x^2 + y^2 + ax + by + c = 0$ into your graphing package.

Your package will make $a = b = c = 1$.

You should see a circle, with centre? Radius?

Now set the values of a, b and c to 0.

(Your circle disappears!)

Now type in all the equations you can get by mixing up a, b and c.

$x^2 + y^2 + ax + cy + b = 0$, $x^2 + y^2 + bx + ay + c = 0$…

How many equations do you have?

Now try varying the values of a, b and c.

How many circles can you get?

What decides how many circles you get?

Risp 15: Circles Or Not?

Teacher notes

Suggested Use: to consolidate/revise circles in coordinate geometry

I have earned a crust in an A Level maths classroom for more years than I dare to remember now, but there are still moments where I am caught out. I was going through a circles-in-coordinate-geometry problem with the whole class the other day, and I found myself saying something I immediately doubted. (This can always be taken as a Risp Alert!) What to do? Explore the glitch in my thinking now, or keep the thread of the lesson going? On this occasion I did the latter, but I did return later to the thought that had discomfited me. And sure enough, there was a risp to be found there.

A few lessons later, I tried the resultant activity with the same group. Prior knowledge? They had met the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$. They had done a little work expanding this and examining the result, going on to classify various second-order curves into circles and non-circles. How confident and flexible was their understanding? This risp would be a good test.

Suppose we start by picking 3, 2 and 1. Then we create the following six equations:

1. $x^2 + y^2 + 3x + 2y + 1 = 0$
2. $x^2 + y^2 + 3x + y + 2 = 0$
3. $x^2 + y^2 + 2x + 3y + 1 = 0$
4. $x^2 + y^2 + 2x + y + 3 = 0$
5. $x^2 + y^2 + x + 2y + 3 = 0$
6. $x^2 + y^2 + x + 3y + 2 = 0$

Your graphing package will give circles for numbers 1, 2, 3 and 6, while Numbers 4 and 5 produce nothing at all. I allowed the group half an hour trying out various triplets for starting numbers, including a triplet of negatives, and a triplet of numbers between 0 and 1. The exploration over for the moment, we left the computers and returned to our classroom, ready to tackle some theory. I drew up two columns, labelled 'Facts' and 'Conjectures', and asked for contributions. The 'Facts' I received were these:

1. You get a circle most of the time.

2. Otherwise you get nothing. If this happens, the largest of your three numbers comes in the end square.

3. Choosing three negative numbers always gives a full set of six circles.

4. Choosing three numbers between 0 and 1 can give six circles, and can give no circles at all.

'Conjectures' were harder to come by. After a lot of thought, one group in effect offered:

"You get a full set of six circles when $0 < a < b < c$ iff $c < a + b$."

Then a second group said:

"If (a, b, c) is a Pythagorean Triple, then you get a full set of six circles."

Where had these come from? My students had not tried to write much down in the computer room. These hypotheses were both based almost purely on their raw mathematical intuition. So were their 'guesses' inspired hits or wild misses? The time had come to introduce the theory as I saw it.

Given $x^2 + y^2 + ax + by + c = 0$, we can rewrite this as:

$x^2 + ax + y^2 + by + c = 0$. Completing the square:

$(x + a/2)^2 - a^2/4 + (y + b/2)^2 - b^2/4 + c = 0$.

So $(x + a/2)^2 + (y + b/2)^2 = (a^2 + b^2 - 4c)/4$.

So this is (maybe) a circle with centre $(-a/2, -b/2)$ and radius $r = (1/2)\sqrt{(a^2 + b^2 - 4c)}$.

So the circle exists iff $a^2 + b^2 > 4c$.

We were able now to examine the truth or falsity of our 'Facts.' If k is negative, the circle always exists, so if a, b, and c are all negative, we get a full set of six circles.

Choosing 3, 2, and 1 gives us only four out of six. But choosing 10, 11, and 12 gives a full set of six circles. If a, b and c are all between 0 and 1, we may get no circles at all. But choosing 1/2, 1/3, and 1/32 gives two circles. Three numbers between 1/2 and 1 always give no circles.

How do our two conjectures look in the light of this? The first, that $c < a + b$ is the condition for six circles, should read $c < (a^2 + b^2)/4$, a good attempt in my book.

And the second? If we have a Pythagorean Triple, then $a^2 + b^2 = c^2 > 4c$, which is true, since the smallest k can be is 5. So the second conjecture is true! Which left me a little amazed. The class were impressed by both conjectures, one along the right lines, the other a theorem, resulting in spontaneous applause for their authors.

Risp 16: Never Positive

Write down a function that is never positive.
Let us call such a function NP.
(NP means "either negative or zero.")
And another.
And another...

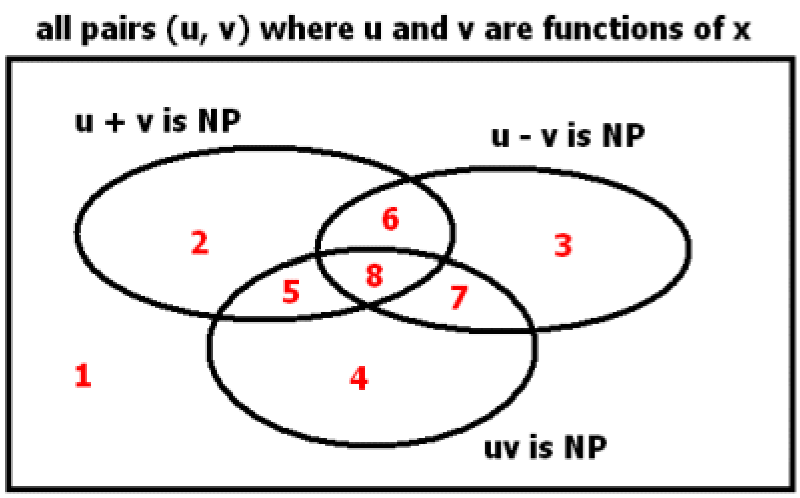
If you add two NP functions, is the result NP?

What if you subtract two NP functions?
Multiply? Divide?

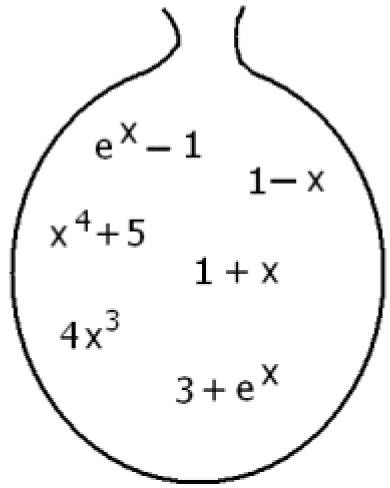
Pick any two functions of x (not just NP ones) and call them u and v.

How can you build an NP function that incorporates both u and v?
Find as many ways as you can.

Try to find a pair of functions (u, v) for each region below.
(Again, u and v do not have to be NP.)



Pick any two functions from the bag below and call them u and v.



Let f(x) = u/v and g(x) = v/u.

Find f '(x) and g '(x).

Show f '(x) g '(x) is NP.

Can you show this will always work, whatever u and v are?

Risp 16: Never Positive

Teacher notes

Suggested Use: to consolidate/revise the Quotient Rule

This is a good example of how a task can be enriched. I wanted to organise some practice for my students on differentiation using the Quotient Rule. [Note: this risp assumes that students know how to differentiate e^x.] One way would be to simply set some exercises from the book, and there are occasions when it would be foolish to do anything else. But given that I had a little more time available, might there be a more oblique way to achieve my goal, by finding an activity that asked for use of the Quotient Rule along the way to examining some wider question?

It seemed natural to me to ask the class to pick two functions of x, called u and v, and then get them to differentiate f(x) = u/v. But if we call g(x) = v/u, we can get two uses of the Quotient Rule for the price of one.

$$f'(x) = (vu' - uv')/v^2, \text{ and } g'(x) = (uv' - vu')/u^2.$$

Can we combine these in some sensible way, so that the result has some interesting property? Multiplying gives:

$$f'(x)g'(x) = (vu' - uv')(uv' - vu')/(v^2u^2) = -(uv' - vu')^2/(v^2u^2).$$

It is clear that whatever u and v are, this will never be positive. (Dividing f '(x) and g '(x) gives the same result.)

Now place this quality, never-being-positiveness, centre stage. I asked my students to play with this notion, and to examine the algebra of NP functions, almost forgetting the initial motivation for the idea as we did this. Now bringing in the Quotient Rule at the end seemed natural and hopefully unforced. Students were practising a target skill within a wider context.

Dinosaur bones or pottery shards alone have little scientific value. The same can said for a student trying to make sense of concepts or procedures. Removed from their contexts, concepts and and procedures lose much of their significance and meaning.

A Handbook on Rich Learning Tasks, 2001,
Gary Flewelling with William Higginson (see Risp Books)

If you add two NP functions is the result NP? For sure. But if you subtract two NP functions, there is no guaranteeing the NP-ness of the result. For example, $f(x) = -2x^2$ take away $g(x) = -x^2$ gives $-x^2$, which is NP. However, $g(x) - f(x)$ gives x^2, which is NN (never negative). Multiplying or dividing two NP functions gives an NN one.

There are lots of ways to combine u and v into an NP function:

$$-|(uv)|, -e^{(u - v)}, -1 + \sin(u + v), \text{etc.}$$

The Venn diagram works nicely here. Is it possible to find (u, v) for the central region? If u + v is NP and so is u - v, then their sum will be NP. Thus u is NP. Since uv is NP, v must be AP. How about: $u = -2x^2$, $v = x^2$? Or even more simply, (0, 0). Answers for the other regions are:

1. (2, 1)
2. $(-1, -x^2)$
3. (x, x)
4. (-2x, x)
5. (x, -x)
6. (-2, -1)
7. $(-x^2, 2x^2)$

The bag of functions is there to stop students from picking over-simple functions for the last part, thus removing the need for the Quotient Rule.

Risp 17: Six Parabolas

Type the equation $y = ax^2 + bx + c$ into your graphing package.
Your graphing package will read a, b and c as variables, and gives them all the value 1 to start with.

Now mix up a, b and c to give all possible orders.
(For example, $y = bx^2 + cx + a$ is one possible order.)
You will get a further five equations:
type these into your graphing package too.
Your graphing package will plot $y = x^2 + x + 1$ each time.

Now try varying a, b, and c using the Constant Controller.
You should find you have six different parabolas if a, b and c are all different.
Do these six parabolas have anything in common?

Can you find a, b and c so that:
1. all the lines of symmetry are to the left of the y-axis?
2. all the lines of symmetry are to the right of the y-axis?
3. all six parabolas have two solutions for $y = 0$?
4. none of the six parabolas have a solution for $y = 0$?
5. exactly one of the parabolas touches the x-axis?

What is special about the vertical lines $x = 0$ and $x = -1$?
Can you explain this?

Show that if a, b and c are rational,
then any point where a pair of parabolas cross has rational coordinates.

Risp 17: Six Parabolas

Teacher notes

Suggested Use: to consolidate/revise quadratic curves

This is a risp that relies on technology, more specifically on the Constant Controller facility in your graphing package. In a sense we create a little microworld with these six parabolas, and their three parameters a, b and c. Prior knowledge about quadratic curves is extremely helpful in tackling the questions posed here successfully, but for those whose theory is skimpy, there is the chance to simply play. Perhaps this is the nearest thing to a computer game that this site has produced so far.

What do the parabolas have in common? Well, a point for starters. (1, k) for some value k lies on all the curves, and one would hope most students can swiftly twig that k is a + b + c and why.

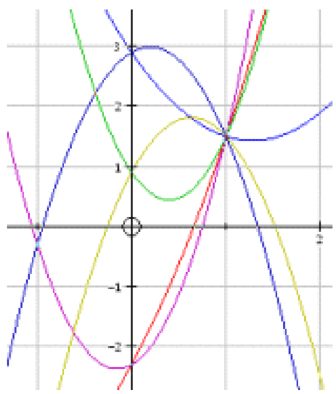


Fig. 1

Managing a, b and c so that all six lines of symmetry are to the left of the y-axis is easy. (In fact, the starting position gives a trivial case of this.) Getting all the lines of symmetry to lie to the right of the y-axis proves infuriatingly difficult, and a little thought combined with some theory reveals this to be impossible. We need $-a/2b$, $-b/2c$ and $-c/2a$ to all be positive. But if a and b are of different sign, and b and c are of different sign, then c and a must be of the same sign, so $-c/2a$ must be negative.

Can we arrange things so that all six parabolas have two solutions for $y = 0$? We can. We need $a^2 > 4bc$, $b^2 > 4ac$, $c^2 > 4ab$, and choosing c to be large and negative means this is certainly possible.

Fig. 2

No solutions? Now we need $a^2 < 4bc$, $b^2 < 4ac$, $c^2 < 4ab$. Clearly a, b and c need to be all positive or all negative. (1, 2, 3) fails, but (1, 1.5, 2) succeeds. (-1, -1.5, -2) will do just as well. (In fact, the original starting position will do, but let us hope our students will have forgotten about that by now!)

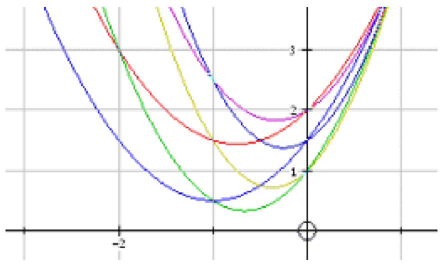


Fig. 3

For a parabola to touch the x-axis, we need, say, $b^2 = 4ac$. But this gives two parabolas that touch the x-axis, $y = ax^2 + bx + c$, and $y = cx^2 + bx + a$. The closest we can get to a single parabola touching the x-axis is if a = c. Should two parabolas on top of each other count as a single parabola? An interesting discussion point.

The lines x = 0 and x = -1 have the property that three pairs of parabolas always seem to intersect upon them. (See Fig. 3) The algebra behind this is pleasing. Suppose we take $y = ax^2 + bx + c$. Where does this intersect with:

i) $y = ax^2 + cx + b$? At $(1, a + b + c)$.
ii) $y = bx^2 + ax + c$? At $(0, c)$ and $(1, a + b + c)$.
iii) $y = bx^2 + cx + a$? At $((a - c)/(a - b), \ldots)$ and $(1, a + b + c)$.
iv) $y = cx^2 + ax + b$? At $((b - c)/(a - c), \ldots)$ and $(1, a + b + c)$.
v) $y = cx^2 + bx + a$? At $(-1, a - b + c)$ and $(1, a + b + c)$.

It is clear from this that if a, b and c are rational, then the coordinates of crossing points will be rational too.

Think of a property of quadratic curves that you would like to test or explore, then think of an aspect of this risp that could generate, examine, require this property in some way. My feeling is that this is a genuinely rich situation, one where many more interesting questions could arise quite naturally.

Risp 18: When does fg equal gf?

Pick two functions, and call them f(x) and g(x).

If fg means "f composed with g", so that fg(x) = f(g(x)), find fg(x) and gf(x).

Does fg(x) = gf(x) for all values of x?

Does fg(x) = gf(x) for any values of x?

Can you think of any f and g so that fg always equals gf?

Pick f so that the graph of y = f(x) is a straight line. Can you find a straight line graph y = g(x) so that fg = gf?

Can you interpret this geometrically?

If f(x) = (ax+1)/(x+b), and g(x) = (cx+1)/(x+d), when does fg = gf?

Investigate other pairs of functions where fg = gf.

Risp 18: When does fg equal gf?

Teacher notes

Suggested Use: To consolidate/revise the composition of functions

Like many, maybe most, of these risps, this one came about mid-lesson, and was suggested by a student comment. Asking when fg = gf is the most natural of questions, but in many years of teaching composition of functions, I had never explored it. One of the marvellous things about teaching mathematics is that whatever the level you teach at, there are always new ways to look at old ideas, and you can always trust your students to find these new ways.

Suppose we choose $f(x) = x^2 + 1$, and $g(x) = 2x + 4$.
So $fg(x) = (2x + 4)^2 + 1 = 4x^2 + 16x + 17$,
and $gf(x) = 2(x^2 + 1) + 4 = 2x^2 + 6$.

So fg(x) does not equal gf(x) for all values of x.

Can we find any x so that $2x^2 + 6 = 4x^2 + 16x + 17$?
This gives $2x^2 + 16x + 11 = 0$, which gives $x = -0.760$ or -7.24.
So fg(x) = gf(x) for two values of x.

If f = g, then fg = gf, whatever f and g are.
If f(x) = x, then fg = gf whatever g is.
If $f(x) = x^n$ and $g(x) = x^m$, then fg always equals gf.

Asking whether fg = gf when both f and g are linear seems like the kind of question that will lead to a trivial solution. But no:

If $f(x) = ax + b$, and $g(x) = cx + d$, then
$fg(x) = a(cx + d) + b = acx + ad + b$,
and $gf(x) = c(ax + b) + d = acx + bc + d$.

So fg = gf implies $ad + b = bc + d$ implies $d = b(c - 1)/(a - 1)$.

So if we choose a, b and c to be anything (excepting a = 1), we can find a value for d that will mean fg = gf. Where do y = f(x) and y = g(x) intersect in this case?

$ax + b = cx + b(c - 1)/(a - 1)$ gives $x = b/(1 - a)$, $y = b/(1 - a)$,
in other words, the two lines intersect on the line y = x.

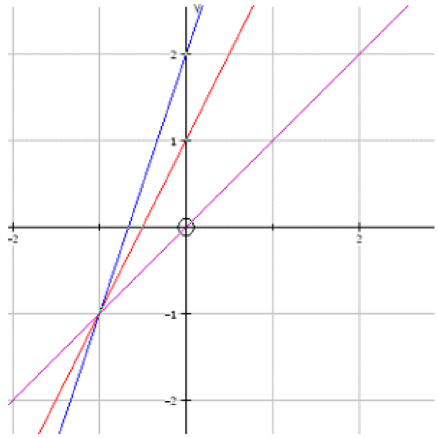


Fig. 1

This gives us the following 'theorem:'

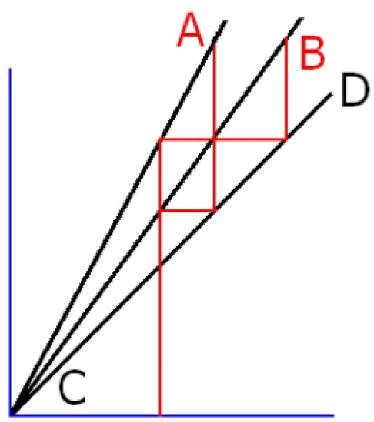


Fig. 2

If the line CD is of gradient 1, then the line AB will be horizontal. Is this as completely obvious as it looks? The link can here be made with some of the numerical methods work on the A2 syllabus.

Now comes the chance to get the stronger students to use their imagination to find other functions that commute.

If $f = (ax + 1)/(x + b)$ and $g(x) = (cx + 1)/(x + d)$, then

$fg(x) = ((ac + 1)x + (a + d))/((b + c)x + (bd + 1))$

and $gf(x) = ((ac + 1)x + (b + c))/((a + d)x + (bd + 1))$.

So $fg = gf$ if and only if $a + d = b + c$, or $a - b = c - d$.

This can lead on to some very deep questions. Given two mobius transformations f(z) and g(z), when does f(g(z)) = g(f(z))? Advanced stuff indeed. The discussion of when two functions commute is meat and drink to group theorists. I am grateful to Ian Short for supplying the insights below:

The question, 'When does fg = gf?' depends on how f and g are chosen. If they are members of a group, then we should use group theory to aid our study. For example, let us work within a group G. Given f in G, the set of all elements g in G that satisfy fg=gf together form a subgroup of G, called the centraliser of f. In other terms, $g^{-1}fg = f$*, so the elements g are the elements that conjugate f to itself.*

A concrete example: consider the symmetries of a regular n-gon. These consists of rotations and reflections. Let r be a rotation that is not the identity map. This commutes with all other rotations (i.e. fr=rf for every other rotation f). On the other hand, for any reflection f, we have that $frf=r^{-1}$*, so f and r only commute if* $r=r^{-1}$ *(that is, r is a rotation by 180 degrees).*

Risp 19: Extending the Binomial Theorem

Pick an odd number greater than 1, and call it n.

Take the numbers 1, -1, and n,
and place them into the square below in some order. (No repeats!)

$$\left(\square + \square x\right)^{\frac{1}{\square}}$$

How many orders are there?

Write down an expression for each order.

Find the first two terms of the expansion of each expression.
(That is, find the constant term and the term in x.)
Assume x is small enough to ignore terms in x^2 and higher.
When will all your expansions be valid at the same time?

Now add your expansions together to give, let's say, A + Bx.

Work out A/B + n. What do you get?

Will this always work? Can you prove it?

Risp 19: Extending the Binomial Theorem

Teacher notes

Suggested Use: to consolidate/revise the Binomial Theorem (for n rational/negative)

Once again, the "How many ways can we mix three numbers up?" technique is employed here. I am finding that my students are beginning to recognise this kind of exercise now, and explanations are no longer needed. They worked with purpose on this risp, aided by the fact that it has a direction of its own. The result it heads towards is not deep (at least I don't think it is!), but at least there is a result to head towards. There is a thought I like from Gary Flewelling and William Higginson (see Risp Books): "A conventional exercise asks you to write a sentence, while a rich learning task invites you to write a story." This risp tells a short story quietly and without histrionics (you could complete this in ten minutes on a good day.)

Another aspect to this task that I like: there are moments of rest built into it. $(2 - x)^1$ can be expanded into a + bx with ease, so all of a sudden being asked to carry out six binomial expansions doesn't look too bad. There is an element of ebb and flow. Everybody can do some of this activity with success.

Suppose I choose 31. (Another nice facet to this activity is that big numbers don't complicate the calculations too much, although they do make them look more impressive!) Expanding as far as x in each case:

$$(1 + 31x)^{-1} = 1 - 31x + \ldots$$
$$(31 + x)^{-1} = (1/31)(1 - x/31 + \ldots) = 1/31 - x/31^2 + \ldots$$
$$(-1 + 31x)^1 = -1 + 31x$$
$$(31 - x)^1 = 31 - x$$
$$(1 - x)^{1/31} = 1 - (1/31)x + \ldots$$
$$(-1 + x)^{1/31} = -1(1 - x)^{1/31} = -1 + (1/31)x + \ldots$$

These will all be valid at the same time if $|x| < 1/31$.

Adding these gives $A + Bx = (31 + 1/31) + (-1 - 1/31^2)x$.

So $A/B + n = (31 + 1/31)/(-1 - 1/31^2) + 31 = 0$.

Running through this with a general odd number n rather than 31 shows that we will always get 0. Thus this risp has the self-checking properties that we have already noticed in others.

Some might say, expanding as far as the term in x is not a real test of whether the Binomial Theorem has been fully understood. I would agree, but this risp is at least a start in that direction. The way the expansion is simplified here may make the activity more accessible for weaker students. And this risp certainly tests the technique for expanding $(a + bx)^c$. Expanding further could make a sensible extension for stronger students. If the expansions add to $A + Bx + Cx^2 + ...$, then $A^3B^2C + B^5 = A^5$. If you can find a nicer relation between later coefficients, please let me know!

I am grateful to Sue Cubbon for sending in the following comment:

The group had commented in passing that they thought they had forgotten what 'Binomial' was about. I was delighted with the consolidation the risp offered, and their excited response to the result. They were determined to prove the result and very pleased with themselves when some could.

This is exactly that response that a risp is designed to elicit.

Risp 20: When does $S_n = u_n$?

Given the sequence u_1, u_2, u_3...
let us say the sum of the first n terms is S_n.

Consider the sequence 11, 9, 7, 5, ...

When does S_n equal u_n for this sequence?
Find all values of n so that this is true.

Still with arithmetic sequences:

If you pick any natural number n, and any first term a,
can you always find a common difference d so that $S_n = u_n$?

If you pick any natural number n, and any common difference d,
can you always find a first term a so that $S_n = u_n$?

What if we look at the geometric sequence
defined by its first term a and its common ratio r?
When does $S_n = u_n$ here?

Take the sequence a, a + d, a + 2d, ...
and the sequence b, br, br^2, ...
and add them to give the sequence a + b, a + d + br, a + 2d + br^2,...
can $S_n = u_n$ with n, r, a, b and d all integers?

Experiment with other sequences/series:
when can $S_n = u_n$?

Risp 20: When does $S_n = u_n$?

Teacher notes

Suggested Use: to introduce arithmetic series/sequences + sequences generally

One embarks upon a long calculation with lots of algebra. Is this worth doing? Will things just become more complicated? We are tempted to move on to something more appealing, when suddenly things begin to open up. Terms in abc cancel, it transpires that pair of terms have factors in common, and then larger factorisations become possible, at which point factors cancel, and we are left with something incomparably simpler than our starting expression. We sense that maybe here lies mathematics that is worth remembering. (As Hardy once said, "There can be no permanent place in the world for ugly mathematics.") The way our algebra has simplified can be an example of such beauty, leaving us perhaps with a sense of awe. The simplification acts as a reward, reassurance that we have chosen the right path and made no mistakes. We have all had that feeling as mathematicians at some time, and it can be enormously satisfying, an Aha! moment. It is an experience I am keen to pass on to my students. This risp has a couple of small Aha! moments, which will hopefully give them the taste for more.

Students who initially know nothing about arithmetic sequences can embark straightforwardly on this activity. The problem is simple: can $S_n = u_n$? Experimentation is the best way to start, and tables can be drawn up for S_n and u_n. For the example I give, we have two possible solutions, n = 1 (which will always be a solution) and n = 13. It is possible to draw a graph of S_n against n, and another on the same axes of u_n against n. The former appears as a parabola, suggesting a quadratic formula, while the latter gives a straight line. There will clearly be at most two intersection points in this situation.

Time for a more general treatment, now that students are ready to look at arithmetic sequences more abstractly, and I can introduce standard terminology and the formula for the sum of the first n terms. This risp provides an immediate use for this.

We need $a + (n - 1)d = n(2a + (n - 1)d)/2$.

This rearranges to $n^2[d] + n[2a - 3d] + [2d - 2a] = 0$.

And now our minor Aha! moment: this factorises (deep down we know it must, as n - 1 must be a factor.)

So we arrive at $(n - 1)(dn + (2a - 2d)) = 0$.

Thus $n = 1$, or $n = 2(d - a)/d$.

This last relation rearranges to $n = 2 - 2a/d$, or $d = 2a/(2-n)$, or $a = d(2-n)/2$.

Thus given a value for n and one for a, we can find a value for d so that $S_n = u_n$, and given a value for n and for d, we can find a value for a so that $S_n = u_n$.

Reflecting upon this, $S_n = u_n$ for an arithmetic series in the 'obvious' places. $n = 1$ will always work, and if terms cancel out exactly (as they do in the 11, 9, 7... example), then there will be a second value of n so that $S_n = u_n$.

What about geometric series? I might offer a practical example at this point, and invite experimentation: can S_n ever equal u_n here?

Once again, the theory has been motivated. Can $ar^{n-1} = a(r^n - 1)/(r - 1)$ (with r not equal to 1)? This gives that $r^{n-1} = 1$, so the only possibility is n odd and $r = -1$. This gives the only solution as the again 'obvious' one, a, -a, a, -a, a,... , when $S_n = u_n$ if n is odd.

So what of this strange hybrid sequence formed by adding an arithmetic and a geometric one? Does it look at all promising? Treating it generally, we need:

$$a + (n - 1)d + br^{n - 1} = n(2a + (n - 1)d)/2 + a(r^n - 1)/(r - 1).$$

And so, embarking upon the attempt to see if this will simplify happily, we get:

$$2ar - 2a + dnr + 2d - nd - 2dr - br^{n-1} - 2anr - dn^2r + 2an + dn^2 + b = 0.$$

Factorising (that little Aha! moment again):

$$2a(r - 1)(n - 1) + d(n - 1)(n + 2)(r - 1) + b(r - 1)(1 + r +...+ r^{n-2}) = 0.$$

$$\text{So } a = b(1 + r + ... + r^{n-2})/(2(1 - n)) - d(n + 2)/2.$$

Thus if d is even and b is a multiple of $2(n - 1)$, a will be an integer too.

Risp 21: Advanced Arithmogons

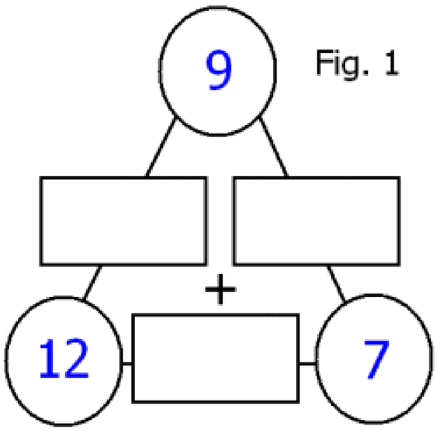


Fig. 1

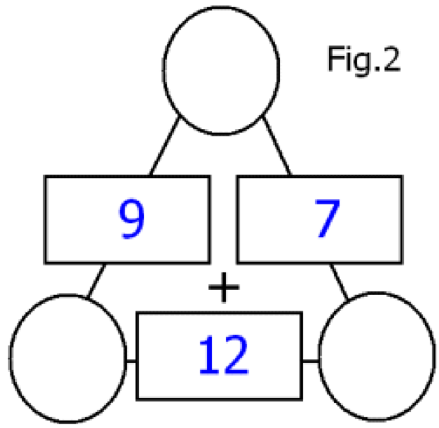


Fig.2

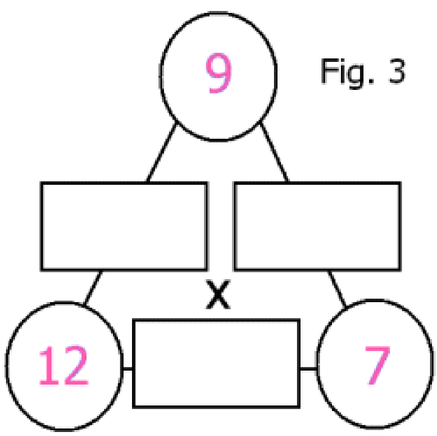


Fig. 3

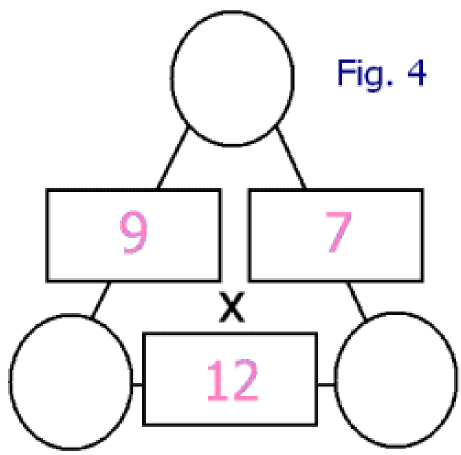


Fig. 4

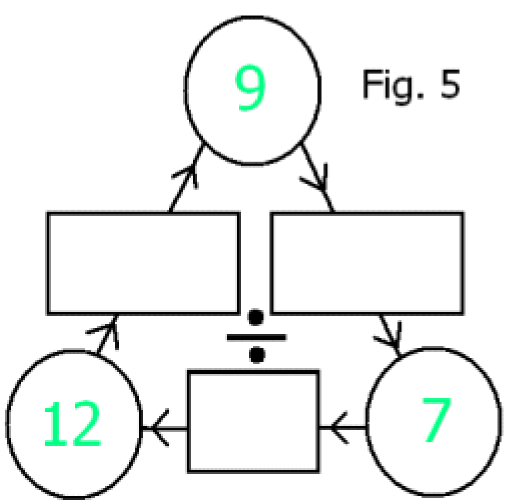


Fig. 5

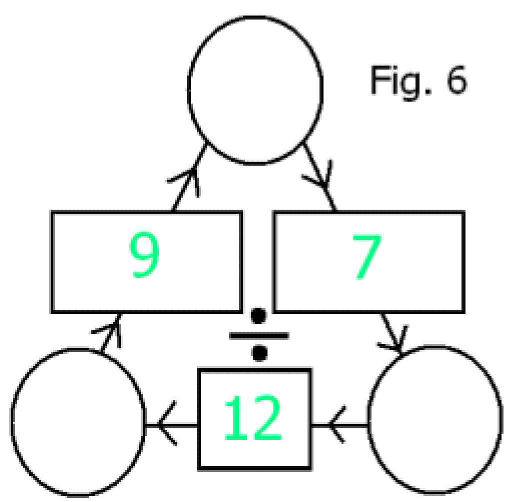


Fig. 6

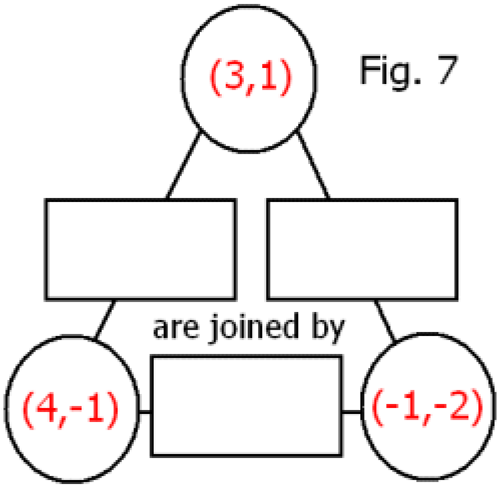


Fig. 7

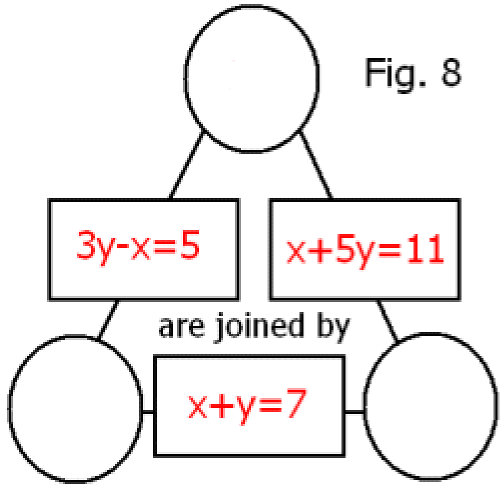


Fig. 8

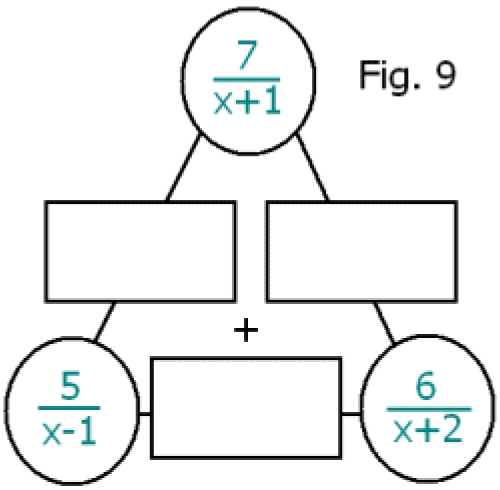


Fig. 9

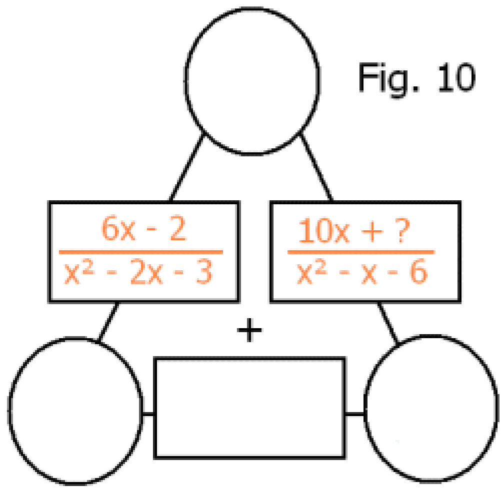


Fig. 10

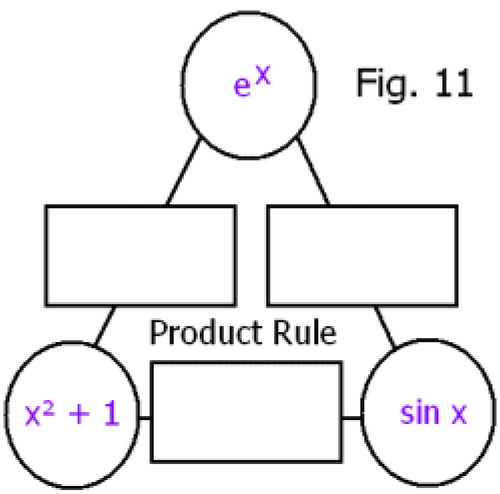


Fig. 11

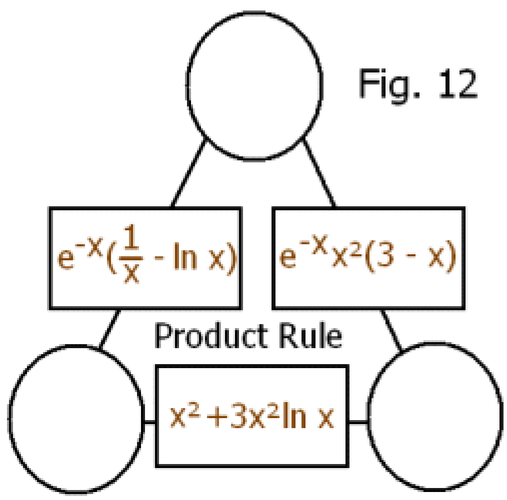


Fig. 12

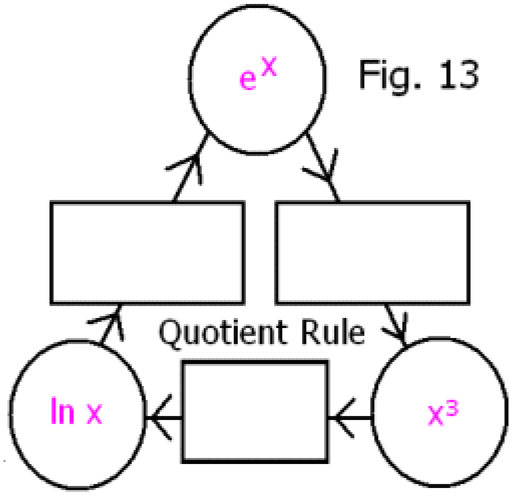


Fig. 13

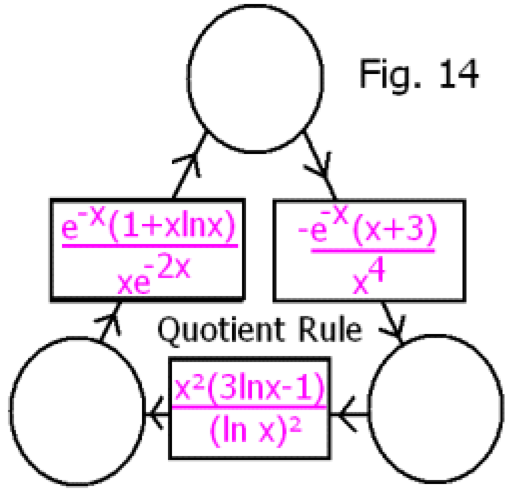


Fig. 14

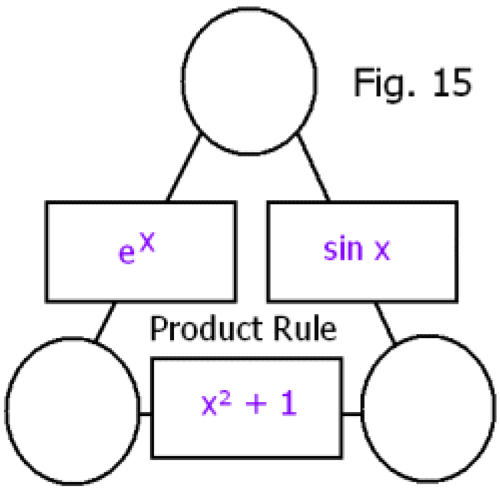


Fig. 15

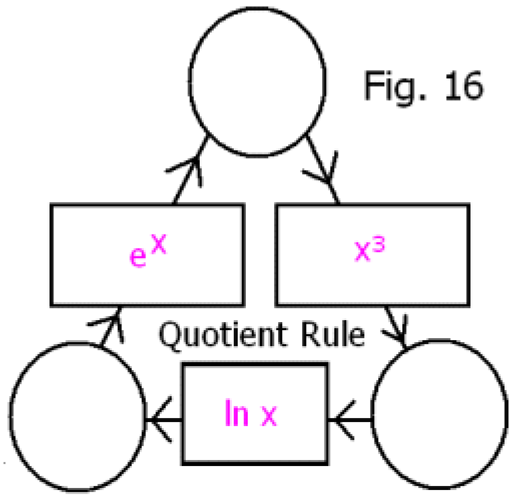


Fig. 16

Risp 21: Advanced Arithmogons

Teacher notes

Suggested Use: To introduce/consolidate/revise ideas about anything!

The human race often seems to be good at doing things, but less good at undoing them. We can do nuclear power, but can we undo a nuclear power station? This may well prove to be a lot harder and more costly than we anticipate. We can pump carbon dioxide into the atmosphere, no problem, but reducing the level of carbon in the air? And we seem quite good at making other living things extinct, but bringing a dodo back from the dead so far eludes us.

So it is within our classroom. Given a rule to work with, a student may be able to follow it successfully, but as soon as some reversal is needed, some mirroring of the rule that requires understanding, they feel confused. If my students can successfully do <u>and</u> undo, then I can be confident that they feel significantly at home with the material.

There is no better way to present ideas of doing and undoing than arithmogons. They have been around a long time: 1975 is the first reference I have for them, but I daresay they have been around longer than that. One of their many marvellous aspects is that they require few words. (I have tried to manage with none at all in presenting this risp!) John Mason and Sue Johnston-Wilder (see Risp Books) give a long and fascinating study of how simple arithmogons might be used in a variety of ways.

Figure 1 is the simplest possible example. Add two circles and put the answer in the box between them. The 'undoing' version is harder, however. If we call the unknowns a, b and c, $a + b = 7$, $b + c = 12$, $c + a = 9$. Simultaneous equations (will your students have met three equations in three unknowns before?) giving $a = 2$, $b = 5$, $c = 7$. Can we find a solution whatever the numbers in the boxes are, and will there always be just one solution?

The multiplication example is a touch more advanced. Here $ab = 7$, $bc = 12$, $ca = 9$, giving $a^2 = 21/4$. So two solutions this time, $a = \pm (\sqrt{21})/2$, $b = \pm 2 (\sqrt{21})/3$, $c = \pm 6(\sqrt{21})/7$. In my experience this is a really good exercise for A Level students early in their course. It is an excellent chance to run over the theory of surds, for example that $\sqrt{(a/b)} = (\sqrt{a})/(\sqrt{b})$, rationalising fractions with a surd in the denominator and so on.

Figures 5 and 6 are directed arithmogons, for the case when the 'doing' is not commutative. Figure 6 in particular raises some interesting questions. It looks harmless enough, but how many solutions does it have? The answer is none, so what is the condition on a, b and c for there to be solutions?

Figures 7 and 8 show how an arithmogon can be extended into any topic you care to name. Rather than set exercises from the book, these figures provide practice with a purpose. My students seem to find the sense of completeness arising from filling the boxes and circles an aid to learning in itself.

Figures 9 and 10 move into partial fractions, a classic doing/undoing scenario. In Figure 10, it is remarkable how much information it is possible to rub out for the problem still to remain viable, providing more useful practice with simultaneous equations. The solutions here are $4/(x - 3)$, $2/(x + 1)$ and $6/(x + 2)$.

Figures 11 - 16 address differentiation and the Product/Quotient Rules. Should you need the solutions, Figures 12 and 14 have e^{-x}, x^3 and $\ln x$ as answers. Figure 15 is fairly possible, I think, but Figure 16... If anyone out there has a solution, I will publish it happily!

Risp 22: Doing and Undoing the Binomial Theorem

$$\frac{(ax+b)^2}{x+c} = p + qx + rx^2 \ldots$$

Pick three different positive numbers for a, b and c.
Find p, q and r.

Pick three different positive numbers for p, q and r.
Find a, b and c.

For what values of x are your expansions valid?

What happens if your starting numbers don't have to be positive?

Risp 22: Doing and Undoing the Binomial Theorem

Teacher notes

Suggested Use: to consolidate/revise the Binomial Theorem (negative/fractional index)

While on the subject of doing and undoing, let's include this risp that once again practices a lot of useful algebra for those who persevere to the end. The 'doing' aspect is straightforward here, but the 'undoing' carries an air of mystery. Will there always be an answer? Will there ever be more than one? As a challenge, it seems a logical, neat problem that deserves an answer. Once again, the algebra treats us kindly along the way, which is always a good sign. First the doing:

$$(ax + b)^2/(c + x) = (a^2x^2 + 2abx + b^2)(1/c)(1 + x/c)^{-1}$$

$$= (a^2x^2 + 2abx + b^2)(1/c)(1 - x/c + x^2/c^2 - ...)$$

$$= b^2/c + x[2ab/c - b^2/c^2] + x^2[a^2/c + b^2/c^3 - 2ab/c^2] + ...$$

This is valid if $|x/c| < 1$.

So the doing is not too much of a problem. Can we undo? (The algebra here is definitely not for the faint-hearted!) I am grateful to Bernard Murphy for pointing out that multiplying by $(x + c)$ is a good idea here, giving:

$$(ax + b)^2 = pc + (p + cq)x + (q + cr)x^2...$$

When using actual numbers rather than tackling the general case, this will be best. Alternatively we can use what we have done above. If (a, b, c) is a solution, then clearly $(-a, -b, c)$ will be a solution too. Let us not count these as different.

$p = b^2/c$, so $c = b^2/p$.

$q = 2ab/c - b^2/c^2$, so substituting for c, and rearranging, we get:

$$a = (b^2q + p^2)/(2pb).$$

$r = a^2/c + b^2/c^3 - 2ab/c^2$, so substituting for a and c and rearranging, we get:

$$b^4[q^2 - 4pr] + b^2[-2p^2q] + p^4 = 0.$$

This is a quadratic in b^2, and the Formula gives:

$b^2 = (p^2q \pm \sqrt{4p^5r})/(q^2 - 4pr)$

^ $b^2 = p^2/(q + 2\sqrt{pr})$ or $p^2/(q - 2\sqrt{pr})$.

So if p and r are of different sign, there will be no real solution.

But if p and r are of the same sign
and $q > 2\sqrt{pr}$, there will be two different solutions for b.

So if (p, q, r) = (2, 17, 8), solutions for (a, b, c) are
(13/3, 2/3, 2/9) and (21/5, 2/5, 2/25).
This means that $(13x + 2)^2/(9x + 2)$ and $(21x + 2)^2/(25x + 2)$
both have binomial expansions that begin
$2 + 17x + 8x^2$... which is surprising, is it not?

Risp 23: Radians and Degrees

Luke was working on a problem that required the use of an angle lying between 45 degrees and 50 degrees.

The sine of this angle had to be calculated.

Unfortunately, Luke forgot that his calculator was in radians mode.

Fortunately, his calculator still gave the right answer.

What was the angle?

What is the nearest angle to this that gives the same sine in degrees or radians <u>and</u> the same cosine in degrees or radians?

Risp 23: Degrees and Radians

Teacher notes

Suggested Use: to consolidate/revise degrees/radians and general trigonometric equations

There are many ways to present this activity. You could gently pick on the member of your class who has the reputation for being most scatty. The truth is we have all initiated calculations that have failed because we have worked with a calculator in radians rather than degrees or vice versa. This risp imagines the happy coincidence when we get away with it.

So we need to solve $\sin x^{o} = \sin x^{c}$.

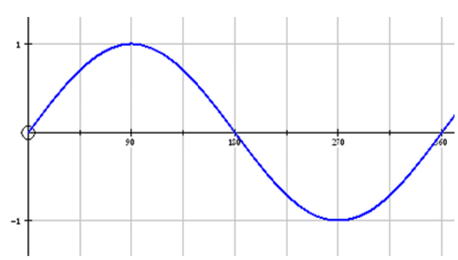
We know that $1^{c} = 180^{o}/\pi$, so we seek to solve $\sin x^{o} = \sin (180x/\pi)^{o}$.

The general solution is $x = 180x/\pi + 360n$, or $x = 180 - 180x/\pi + 360n$.

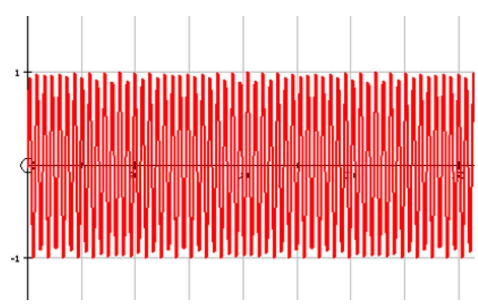
These solve to give $x = 360n\pi/(\pi - 180)$ [Formula 1] or $(180 + 360n)\pi/(\pi + 180)$ [Formula 2].

Now it is a question of trying out values for n to see if we can get a value for x between 45 and 50. This happens only if we put n = 7 in Formula 2, giving x = 46.31553129... $^{o/c}$.

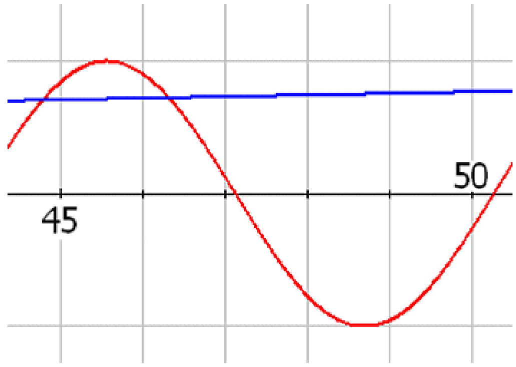
It is nice to tackle this graphically. The graph of y = sin x looks like this:



But change degrees to radians here and we get this:



Plotting these together gives lots of crossing points, but only one in [45, 50].



What if we insist also that cos x^o = cos x^c?

This gives x = 180x/π + 360n, or -180x/π + 360n. These solve to give:

x = 360nπ/(π - 180) [Formula 1] or 360nπ/(π + 180) [Formula 3].

So Formula 1 will always give a value x so that sin x^o = sin x^c and cos x^o = cos x^c.

Which of these values is closest to 46.31553129...?
n = -7 gives x = 44.76356881... as shown on the graph above.

Risp 24: The 3-Fact Triangles

A 3-Fact Triangle is one where three of the following facts are true.

1. One side is 3cm.

2. One angle is 90°.

3. One side is 4cm.

4. One angle is 30°.

How many 3-Fact triangles are there?

Find the area and perimeter of each one.
Place the triangles in order of size for area and for perimeter.

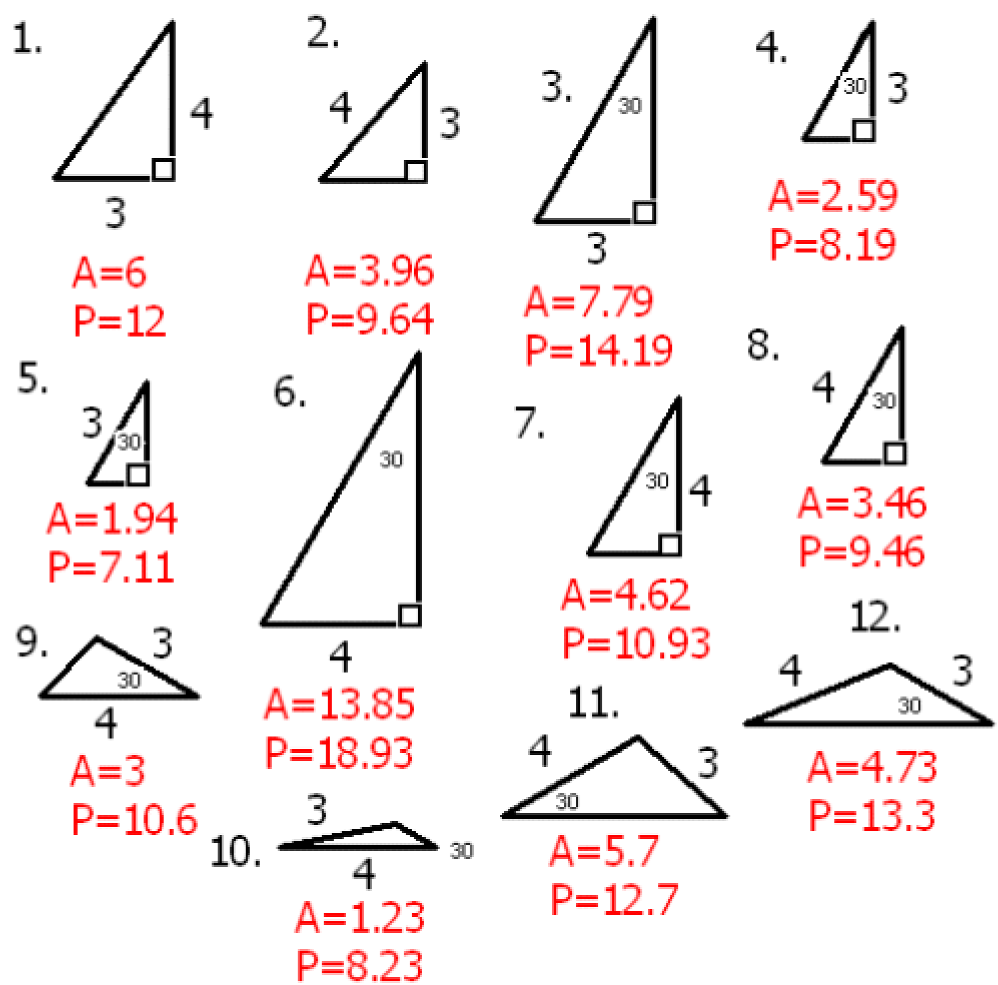
Risp 24: 3-Fact Triangles

Teacher notes

Suggested Use: to consolidate/revise Sin Rule/Cos Rule/Area formula

I will be honest, the Sin Rule and the Cos Rule are topics that I often approach with a sinking heart. They are the sledgehammers in a mathematician's toolkit, useful, indispensible maybe, but not noted for their elegance. There is often too a great variety in prior learning here, so there is a risk that the higher flyers may be left twiddling their thumbs. This risp requests a generous amount of logic alongside the need to put the rules into practice. When I tried it with a group, my students seemed to me to be much more engrossed than if ploughing through standard consolidation. There was enough here to last a full 90 minutes. To get to the end, the Sin Rule is required, including the ambiguous case, so is the Cos Rule, and so is the area formula (1/2)absin C.

There are twelve 3-Fact Triangles:



Of these, the hardest to find are the 10./11. pair. Using the Sin Rule for these two triangles is excellent practice. It is also good to discuss why there is not a similar pair for triangle 12.

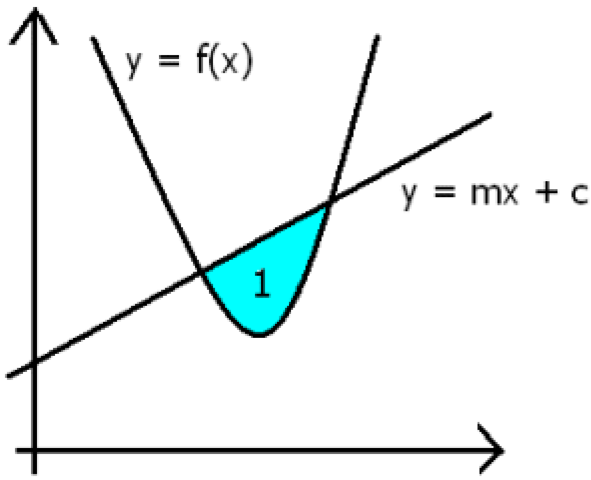
From large to small (Area) the list is:
6, 3, 1, 11, 12, 7, 2, 8, 9, 4, 5, and 10.

From large to small (Perimeter) the list is:
6, 3, 12, 11, 1, 7, 9, 2, 8, 10, 4 and 5.

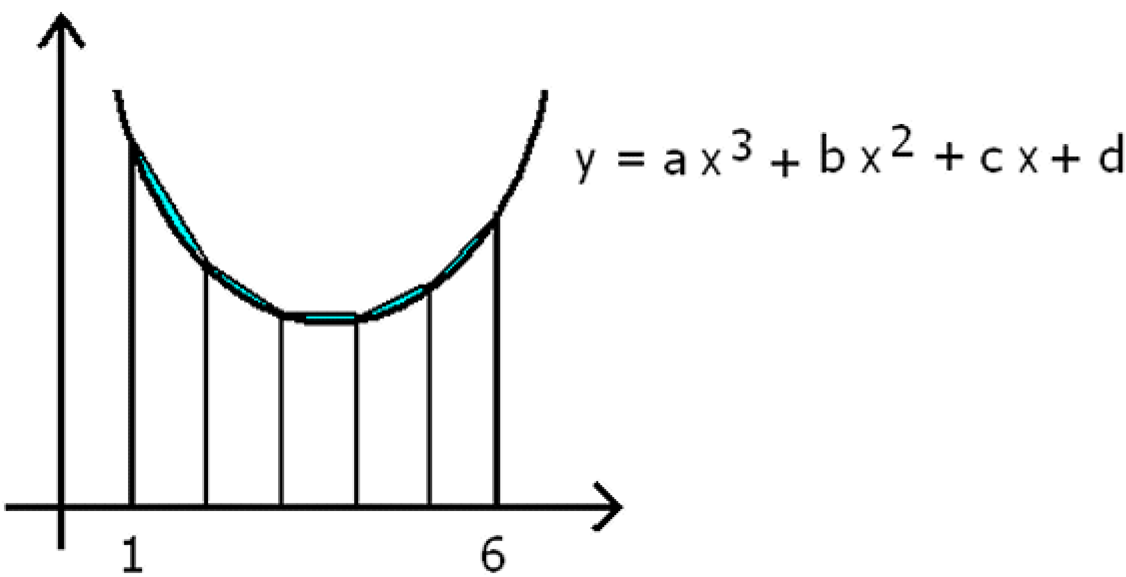
You may wish to examine the ratio Area/Perimeter for each triangle.

This task can be extended in lots of ways. The whole notion of choosing n facts that may be individually true about an object, but which cannot all be simultaneously true, and then examining the possible number of objects where m of them are true seems to me to be fruitful. In this case one could ask, if one gives two facts that specify sides and two facts that specify angles, will there always be twelve 3-Fact Triangles? What is the maximum number of possibilities? The minimum?

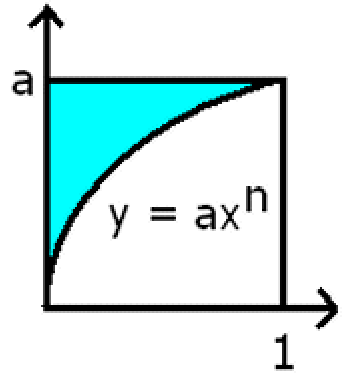
Risp 25: The answer's 1: what's the question?



Find a possible f(x) and mx + c in the diagram above.



In the diagram above, the Trapezium Rule with five strips overestimates the area under the curve between x = 1 and x = 6 by exactly 1 square unit.
Find possible values for a, b, c and d.



If you rotate $y = ax^n$ between 0 and 1 about the x-axis,
you get volume V_1.
If you rotate $y = ax^n$ between 0 and a about the y-axis,
you get volume V_2.
If $V_1 - V_2 = 1$, find possible values for a and n.

Risp 25: The answer's 1, what's the question?

Teacher notes

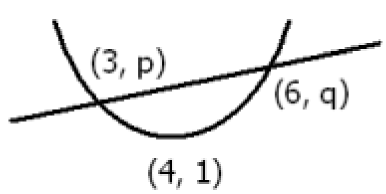
Suggested Use: to consolidate/revise area between two curves, Trapezium Rule, Volumes of Revolution, etc

It is extraordinary how much more interesting a question becomes if is opened out, thereby escaping the tyranny of the find-the-one-and-only-correct-answer approach. Compare the activity "Give the answer to 3 + 4" with "Give a question to which the answer is 7." The first is closed and fails to differentiate, whilst the second encourages a creative response, can be extended ad infinitum and can be answered at any level. It could be argued that this second question is actually a little too open: there is a delicate balance here. A totally open question can be as dull as a narrowly closed one. Perhaps saying "Give questions to which the answer is 7 that include a + sign, a squaring and the digit 4" would be an improvement.

For this risp, students are presented each time with a picture that the topic to be studied might easily generate, but with key constants omitted. We then suppose that the answer to some natural question turns out to be really nice, say the value 1. The challenge is then to find a set of constants that will make this work.

Students may take a little time to get used to this new way of doing things. They may not initially realise that they are free to pick certain values, and that if they do this cleverly they can create a much easier question for themselves. But when they do cotton on, they will appreciate the freedom they have been given, and they will practice those key skills you wish to cultivate whilst having some wider question in view.

The area between two curves is a topic where students who know a bit about integration can be thrown straight in. They might choose the curve in the picture to be a parabola with minimum point (4,1). So its equation is $y = a(x - 4)^2 + 1$. [Students can be encouraged to use this form for the equation of the parabola before starting.]



So $p = a + 1$, $q = 4a + 1$, $m = (q - p)/3 = a$, $c = 1 - 2a$.

So $1 = \int_3^6 ax + 1 - 2a \, dx - \int_3^6 ax^2 - 8ax + 16a + 1 \, dx$

Solving this is the usual way gives a = 2/9. Of course there are lots of other choices that will work just as well.

The Trapezium Rule question is the toughest.
It is possible to find the area under the curve exactly in terms of a, b, c and d.

$A = \int_1^6(ax^3 + bx^2 + cx + d)dx = 1295/4a + 215b/3 + 35c/2 + 5d.$

Using the Trapezium Rule with five strips,

A is approximately 665/2a + 145b/2 + 35c/2 + 5d.
(Notice the terms in c and d are the same here.)

So 1 = 35/4a + 5b/6, so b = (12 - 105a)/10, while c and d can be anything that makes the picture look right. We could choose the minimum to be at (4,1).

$y' = 3ax^2 + 2bx + c$, $y'' = 6ax + 2b$.

So 48a + 8b + c = 0, since y' = 0 when x = 4.

In addition the gradient is increasing from 1 to 6, so y'' > 0 in that range.

A sensible choice for a seems to be 0.1, which gives b = 0.15, c = -6, and d about 20.

The volumes of revolution question works out fairly neatly.

$\int_0^1 \pi y^2dx = \int_0^1 \pi a^2x^{2n}dx = \pi a^2/(2n + 1).$

$\int_0^a \pi x^2dx = \int_0^a \pi y^{2/n} / a^{2/n} dx$

$= \pi an/(2+n).$

So $1 = \pi a^2/(2n + 1) - \pi an/(2 + n)$.

At this point we could choose a = 1, in which case n = 0.478...,
or else maybe n = 1 (a straight line), in which case a = 1.5976...
or else n = 1/2...

Risp 26: Generating the Compound Angle Formulae

Pick two distinct whole numbers between 1 and 10 inclusive, and call them n and m.

Write down the functions sin nx, sin mx, cos nx and cos mx.

Put these functions into the squares below in some order (no repeats!)

$$y = (\square)(\square) + (\square)(\square)$$

How many different ys can you make?

Draw each possible y using your graphing program - do you recognise any of these curves?

Based on what you have done, to what would you guess will the function sin A cos B + sin B cosA simplify?

Now put your four functions into the squares below in some order (no repeats!)

$$y = (\square)(\square) - (\square)(\square)$$

How many different ys can you make this time?

Draw each possible y using your graphing program - do you recognise any of these curves?

Predict other connections between sin A, cos A, sin B and cos B, and test these out.
Can you prove them?

Risp 26: Generating the Compound Angle Formulae

Teacher notes

Suggested Use: to introduce Compound Angle Formulae

Once again I return here to the Explore-by-Mixing-Things-Up method. When it comes to compound angles, this technique almost suggests itself, as we can see its results before our eyes. Suppose we pick n = 4, and m = 3. We get nine possible functions for y altogether: what happens if we plot these?

1. sin 3x cos 4x + sin 4x cos 3x - gives sin 7x
2. sin 3x cos 3x + sin 4x cos 4x - weird!
3. sin 3x sin 4x + cos 3x cos 4x - gives cos x
4. sin 3x cos 4x - sin 4x cos 3x - gives -sin x
5. sin 3x cos 3x - sin 4x cos 4x - weird!
6. sin 3x sin 4x - cos 3x cos 4x - gives -cos 7x
7. sin 4x cos 3x - sin 3x cos 4x - gives sin x
8. sin 4x cos 4x - sin 3x cos 3x - weird!
9. cos 3x cos 4x - sin 3x sin 4x - gives cos 7x

This provides excellent practice in identifying transformations applied to standard trig curves. Once students have come up with possible identities, they can be challenged to prove them. Given that they now know what they heading towards, they will have a much better chance of doing this, although I acknowledge this is tough. But trying first means that when the teacher comes to prove some or all of these, attention is guaranteed!

Risp 27: Playing with Parametric Equations

A curve is defined using the parameter t as follows:

$y = at^2 + bt + 1$.

$x = t + c$.

When $t = 1$, $dy/dx = d$.

When $t = 2$, $dy/dx = e$.

The curve goes through (1, 1).

If $a = 2$ and $b = 3$, find c, d and e.

If $c = 2$ and $d = 3$, find a, b and e.

Given two of a, b, c, d and e, can we always find the others?
How many solutions will there be?
Can you interpret this geometrically?

Risp 27: Playing with Parametric Equations

Teacher notes

Suggested Use: to consolidate/revise parametric equations

Let's jump straight in with the general solution.

$dy/dt = 2at + b$, $dx/dt = 1$, so $dy/dx = 2at + b$.

So $2a + b = d$, and $4a + b = e$.

What value of t gives $(1, 1)$?

$1 = t + c$, so $t = 1 - c$.

So $1 = a(1 - c)^2 + b(1 - c) + 1$.

This gives $(1 - c)(a + b - ac) = 0$.

So $c = 1$ or $(a + b)/a$.

If $a = 2$ and $b = 3$, $c = 1$ or 2.5, $d = 7$, $e = 11$.
So there are two solutions.

If $d = 2$ and $e = 3$, $a = 0.5$, $b = 4$, $c = 1$ or 9.
So there are two solutions.

Given two of a, b, d and e, we can always find c and the others, with c taking two possible values, one being 1. So there will be two solutions.

If we are given c (not 1) and one other from a, b, d, and e, then we can always find the others uniquely.

But if we are given $c = 1$ and $b = 2$, say, we do not have enough information to determine the other constants. So infinitely many solutions exist.

Risp 28: Modelling the Spread of a Disease

A population is threatened by an infectious disease.

Imagine that the population splits into two groups, the infected and the healthy.

Each year, the probability that a healthy person catches the disease is c, and the probabilty that an infected person recovers is r.

Each of you is going to model what happens to 10 people over a period of 10 years.

Everyone starts with 8 healthy people and 2 infected people.
Year 0 on your list is 8 Hs and 2 Is.
We are going to assume that no-one dies (of any cause) over this 10 years.

For each person roll a die, for each year.
If you roll a 1 or 2 for a healthy person,

then they become infected that year (c = 1/3).
If you roll a 1 for an infected person, then they recover that year (r = 1/6).
For everything else, there is no change.

When you have each completed 100 rolls of the dice, we can pool our data, to arrive at the total numbers of H/I people for the population for each year.

Let x = proportion of population that are infected.

What happens if we plot a graph of x against time?
What happens as x tends to infinity?

Can we model this situation with a differential equation?
Can you solve the differential equation?
What happens to the solution if we vary c and r?

Risp 28: Modelling the Spread of a Disease

Teacher notes

Suggested Use: to consolidate/revise differential equations (variables separable only)

Here is a risp with a slightly topical flavour. As I type, the UK and the other nations of the world are bracing themselves to withstand avian flu. Although there is no real threat to humans currently, the danger is that the virus will mutate into a form that can be spread from human to human. Journalists have rekindled memories of the horrendous flu epidemic at the end of the First World War, and we are told that we are due for something similar again. It seems timely then to tackle epidemiology in a simple way within the A Level mathematics classroom.

Do I have a sadistic streak? Maybe sometimes a little hard labour does no harm in a mathematics classroom. In particular, I think there is no substitute in this exercise for actually rolling a dice 100 times and recording the results (this takes 10 minutes at most.) One could flick on an Excel program and ask it to perform this labour in a second, but would this really be appreciated by a group who had not experienced the harder route first? It makes the ten years in the question feel more like ten years. In fact, the rolling can induce a sense of curiosity and purpose into the lesson. The Excel program will come in handy eventually when you start to vary c and r.[It is hoped to add such a program to this site before too long.]

Differential equations do lie within the Core for A2, but only in their simplest form. 'Variables separable' is the only type of DE that our C4 module contains, and fortunately, the DE generated by this scenario lies within this category.

A little reflection shows that $dx/dt = c(1 - x) - rx$.

If $c = 1/3$ and $r = 1/6$ as in our starting example,
$dx/dt = (1/3)(1 - x) - x/6$.

So $dx/dt = 1/3 - x/2$.

Separating the variables, $(6/(2 - 3x))dx = dt$.

So $-2\ln (2 - 3x) = t + a$.

So $x = 2/3 - be^{(-t/2)}$.

When t = 0, x = 0.2, so b = 7/15, and we end with x = 2/3 - 7(e^(-t/2))/15.
As t tends to infinity, x tends towards 2/3. Hopefully this will fit with your data.

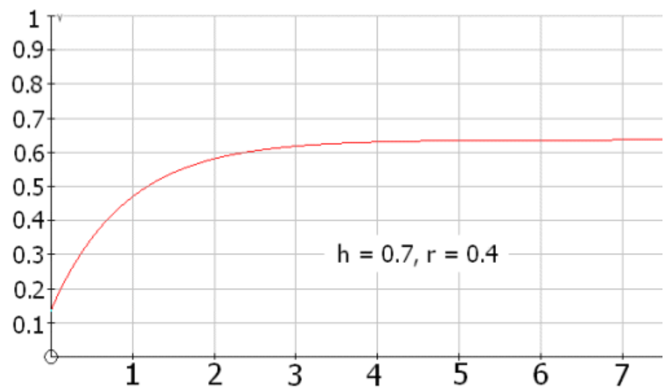
Starting in general from dx/dt = c(1 - x) - rx:

dx/dt = c - (r + c)x.

Separating the variables, (1/(c - (r + c)x)dx = dt.

So (-1/(r + c)) ln (c - (r + c)x) = t + a.

So x = c/(r + c) - be^(-(r + c)t).
This can be entered into Autograph, and the constants can be varied.



As t tends to infinity, x tends to c/(r + c).

If c is big in relation to r, c/(r + c) will be close to 1.
If c = r, then x tends to 1/2.
If r = kc, then x tends to 1/(k + 1).

I would not jump immediately to the general case with my students here. Indeed, this is quite a tough risp to use as an introduction to DEs. It might best be used as the start of the concluding lesson on the subject.

Risp 29: Odd One Out

For each triplet,
try to think of ways in which each member could be the odd one out.

For example, given the triplet 2, 3, 9:

2 could be the odd one out because it is even and the others are odd.
3 could be the odd one out because it is the only triangle number.
9 could be the odd one out because it is composite and the others are prime.

Triplet 1: $\sin x$, $\cos x$, $\tan x$

Triplet 2: e^x, $\ln x$, x^2

Triplet 3: $\sqrt{2}\underline{i}$, $\underline{i} + \underline{j}$, $(1-\sqrt{2})\underline{i} + \underline{j}$

Triplet 4: $\cos 2x$, $\sin 2x$, $\cos x + \sin x$

Triplet 5: x, x^2, x^3

Triplet 6: $(\cos t, 1 + \sin t)$, $(t, t^2 + 3)$, $(t, 3t)$

Triplet 7: $\sqrt{2}(1 - 2x)^2$, $(1 + 2x)^{-1}$, $(2 - x)^{1/2}$

Triplet 8: $\sec^2 x$, -1, $\tan^2 x$

Triplet 9: $\underline{i}$, $\underline{i}.\underline{j}$, $\underline{i}.\underline{j}.\underline{k}$

Triplet 10: $r = \underline{i} + a\underline{j}$, $r = b(\underline{i} - \underline{j})$, $r = \underline{k} + c(\underline{j} - \underline{i})$

Risp 29: Odd One Out

Teacher notes

Suggested Use: To consolidate/revise anything!

Gatsby Teacher Fellows are wisely given a mentor to shepherd them through their project year. I was fortunate enough to be given two mentors (did I look particularly helpless?), Bernard Murphy, leader of the Teaching Advanced Mathematics initiative and CPD Coordinator at MEI, and Susan Wall, ex-Head of Mathematics at Wilberforce College and a major author for the Standards Unit *Excellence for All* materials. Bernard was a welcome visitor in the lesson where I used this Odd One Out risp for the first time. (The Odd One Out technique is explored in *Improving Learning in Mathematics: Challenges and Strategies,* by Malcolm Swan (see Risp Books).

I felt the need to warn Bernard about the group in advance. They fell neatly into the Wise Virgins and the Foolish Virgins. The former were always on time, never missed a homework, and came to extra sessions of their own volition, Whilst the latter would wander in late, lose vital sheets as soon as they were given them, and produce work only after sustained and wearying pressure from myself. I wondered whether the membership of the two groups would be as obvious to Bernard as it was to me.

As my students began work, something marvellous happened. They were ALL drawn in to the activity, they ALL became animated, they ALL began to conjecture and brainstorm and share their knowledge or lack of it, without fear or embarrassment. Perhaps the Wise Virgins were more technically able, but the Foolish Virgins made up for this by being quicker to take risks. I was left reflecting that a good risp is a leveller. Bernard came up to me half way through with a smile on his face. "I can't tell who is who!"

Odd One Out – Possible Answers

Triplet 1:

sin (180 - x) = sin x, cos (180 - x) = -cos x, tan (180 - x) = -tan x
OR sin x is product of the other two, whilst the others are not

cos x is even, whilst sinx and tanx are odd
OR cos (0) = 1, whilst sin (0) = 0, tan (0) = 0

tan x has period = 180, the others have period 360
OR tan x is unbounded while others are bounded functions
OR tan 45 = 1, whilst sin 45 = cos 45 = $1/\sqrt{2}$

Triplet 2:

e^x differentiates to itself, the others don't

ln x can be negative, whilst e^x x^2 are always non-negative
OR ln x has a graph that does not cut the graphs of the other functions
OR ln x is undefined for x negative, whilst the others are

x^2 can have negative gradient, whilst the others always have a positive gradient
OR is not inverse of either of the others
OR goes through the origin whilst the others do not
OR even whilst the others are not
OR many-1 whilst others are 1-1

Triplet 3:

$\sqrt{2}\underline{i}$ has no $\underline{j}$ component

$\underline{i} + \underline{j}$ is sum of the other two

$(1-\sqrt{2})\underline{i} + \underline{j}$ does not have modulus $\sqrt{2}$

Triplet 4:

cos 2x can be written in three ways

sin 2x is 0 when x = 0, the others are both 1

cos x + sin x has period 360, the others have period 180
OR can be bigger than 1, whilst the other functions cannot

Triplet 5:

$y = x$ has constant gradient, the others do not

$y = x^2$ is even, the other two are odd
OR x^2 is many-to-1, the others are 1-to-1

$y = x^3$ has a point of inflection, the other two do not
OR is the product of the other two, whilst the others are not

Triplet 6:

$(\cos t, 1 + \sin t)$ is bounded, $(t, t^2 + 3)$ does not go through the origin, $(t, 3t)$ can go below the x-axis

Triplet 7:

$\sqrt{2}(1 - 2x)^2 = \sqrt{2} - 4\sqrt{2}x + 4\sqrt{2}x^2$ (finite expansion)

$(1 + 2x)^{-1} = 1 - 2x + 4x^2 + \ldots$ (does not start with $\sqrt{2}$)

$(2 - x)^{1/2} = \sqrt{2} - (\sqrt{2}/4)x - (\sqrt{2}/32)x^2 + \ldots$ (x^2 coefficient is –ve)

Triplet 8:

$\sec^2 x$ integrates to an exact trig function

-1 is always negative, the others are always non-negative

$\tan^2 x$ is sum of the other two

Triplet 9:

$\underline{i}$ is a vector, the others are not

$\underline{i}.\underline{j}$ is a scalar, the others are not

$\underline{i}.\underline{j}.\underline{k}$ is meaningless, the others are not

Triplet 10:

$r = \underline{i} + a\underline{j}$ (direction is $\underline{j}$, other two have direction $\underline{i} - \underline{j}$)

$r = b(\underline{i} - \underline{j})$ (goes through the origin, the other two do not)

$r = \underline{k} + c(\underline{j} - \underline{i})$ (does not meet the other two lines, the other two do meet)

Risp 30: How Many Differential Equations?

Pick four whole numbers, and call them a, b, c, and d.

Write down the expression $ax^2 - ax + b$. This is Expression 1.

Differentiate Expression 1 with respect to x. This gives Expression 2.

Expression 3 is $cy + d$.

Place Expressions 1 to 3 in the squares below in some order. (No repeats!)

$$(\square)\frac{dy}{dx} = (\square)(\square)$$

Find the general solution for each differential equation that you can make.

You are given the additional information that (0, 0) lies on every solution.

Show that (1, 0) also lies on every solution.

Risp 30: How Many Differential Equations?

Teacher notes

Suggested Use: to consolidate/revise differential equations (variables separable only)

Writing risps can be a great pleasure. One has a good idea to start with, where the main body of the problem is clear, then the challenge is to come up with a final line that ties the ends together neatly. Here is an example where after weeks of trying possibilities, the punch-line eventually suggested itself. I wanted (0, 0) and (0, 1) to lie on all solutions, and this implied that the quadratic would need to be of the form $ax^2 - ax + b$.

The Mix-Them-All-Up technique produces just three distinct differential equations here, but each requires a different method of solution. There is $\int f(y)dy = \int g(x)h(x) dx$ with f, g and h being polynomials, where it is necessary to multiply out g(x)h(x). Secondly, there is the need to recognise $\int f'(x)/f(x)\, dx$, which gives a natural logarithm. And thirdly, there is the need to integrate g(x)/h(x) where this is improper. Algebraic long division here leads to a polynomial plus a natural logarithm. The idea with students is to address particular cases, but below I tackle the case in general.

Differential Equation 1

$(ax^2 - ax + b)dy/dx = (cy + d)(2ax - a)$ gives $\int(1/(cy + d))dy = \int((2ax - a)/(ax^2 - ax + b))dx$.

So $(\ln|cy + d|)/c = \ln|ax^2 - ax + b| + k$.

(0, 0) on the curve means $(\ln|d|)/c = \ln|b| + k$, so $k = \ln(|d|^{1/c}/|b|)$.

So particular solution is $|cy + d|^{1/c} = (|d|^{1/c}/|b|)|ax^2 - ax + b|$.

It is easy to check that (1, 0) is on the curve.

Differential Equation 2

$(cy + d)dy/dx = (ax^2 - ax + b)(2ax - a)$.

$cy^2 + dy = \int(2a^2x^3 - 3a^2x^2 + a^2x + 2abx - ab)\, dx$.

$= a^2x^4/2 - a^2x^3 + a^2x^2/2 + abx^2 - abx + k$.

(0, 0) is on the curve, so $k = 0$.

So particular solution is $cy^2 + dy = a^2x^4/2 - a^2x^3 + a^2x^2/2 + abx^2 - abx$.

It is easy to check that (1, 0) is on the curve.

Differential Equation 3

$(2ax - a)dy/dx = (ax^2 - ax + b)(cy + d)$.

So $\int(1/(cy + d))dy = \int((ax^2 - ax + b)/(2ax - a))dx$.

$(\ln |cy + d|)/c = \int x/2 - 1/4 + (b - a/4)/(2ax - a)dx$.

$= x^2/4 - x/4 + ((b - a/4)\ln |2ax - a|)/2a + k$.

(0, 0) is on the curve, so $k = (\ln |d|)/c - ((b - a/4)\ln |-a|)/2a$.

So particular solution is:
$(\ln |cy + d|)/c = x^2/4 - x/4 + ((b - a/4)\ln |2ax - a|)/2a + (\ln |d|)/c - ((b - a/4)\ln |-a|)/2a$.

It is easy to check that (1, 0) is on the curve.

Risp 31: Building Log Equations

You are given that a, b and c are all positive numbers.

$\boxed{\log_a}$ $\boxed{\log_b}$ $\boxed{\log_c}$

$\boxed{a}$ $\boxed{b}$ $\boxed{c}$

$\boxed{+}$ $\boxed{+}$ $\boxed{=}$ $\boxed{0}$

Cut out the cards above (or imagine doing this),
and arrange some or all of them into an equation
that obeys the rules for mathematical grammar.
The cards should sit in a line, and you have to include at least one 'log' card!

For each equation, say if it is always true,
sometimes true or never true.

If it is sometimes true,
give an example where the equation works and one where it doesn't.

If the equation is always true or never true,
try to say why.

Risp 31: Building Log Equations

Teacher notes

Suggested Use: to consolidate/revise logarithms

This is a tough but rewarding risp that can be used to revise logs and their properties. The task differentiates well: there are some simple equations that all students can create and comment upon, while it is not hard for stronger students to construct more demanding equations that need a lot of thought. Below are seven possible equations that may come up.

$\log_a a = \log_b b$.

This is always true.

$\log_a a = 0$.

This is never true.

$\log_a b = 0$.

This is sometimes true (if $b = 1$).

$\log_a b = \log_b a$.

Say $\log_a b = \log_b a = k$. Then $a^k = b$ and $b^k = a$, so $a^{k2} = a$, so $k^2 = 1$. Thus $k = 1$ or -1, so this equation is sometimes true. There is the obvious solution here of $a = b$, but additionally $\log_a(1/a) = \log_{(1/a)}a$, which it is good for students to spot.

$\log_a b + \log_b a = \log_c c$.

We could write this as $x + y = 1$, where $a^x = b$ and $b^y = a$, so $a^{xy} = a$. So we need to solve $x + y = 1$ with $xy = 1$, and these equations have no common solution. So this is never true.

$\log_a b \log_b a = \log_c c$.

Using the same technique as above, this is always true. [With thanks to Bernard Murphy.]

$\log_a \log_b b = 0$.

The only way of reading this sensibly gives $\log_a 1 = 0$, which is always true.

$\log_a \log_b \log_c c = 0$.

Similarly, this is never true.

$\log_a b + \log_b c + \log_c a = 0$.

This could be written as $x + y + z = 0$ (surface 1), with $a^x = b$, $b^y = c$ and $c^z = a$, which gives $a^{xyz} = a$, or $xyz = 1$ (surface 2). It is helpful to draw this pair of surfaces in 3-D Autograph, and you can see the intersection points. Alternatively we need $x + y + 1/(xy) = 0$, which provides plenty of possibilities. So sometimes true.

Risp 32: Exploring Pascal's Triangle

What do you make of the following argument?

"We know that $^5C_2$ is 10, and 5 times 2 is 10. So ^nC_r is equal to n times r."

Are there any other (n, r) pairs that give $^nC_r = nr$?

When is ^nC_r closest to being nr?

Risp 32: Exploring Pascal's Triangle

Teacher notes

Suggested Use: to consolidate/revise Pascal's Triangle, Binomial Coefficients

One small comment in the classroom can give birth to a substantial risp. I am grateful to Colin Foster, editor of *Mathematics Teaching*, for relaying to me the remark from one of his students that begins this activity: "${}^5C_2 = 10$, so ${}^nC_r = nr$." If only things could always be so simple. Certainly there are other examples where this holds: ${}^nC_1 = n$, so we have an infinite set of examples here. It seems a good idea to draw Pascal's Triangle and to mark in with a red dot where nr overtakes ${}^nC_r$, as eventually it must.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

It is a small step from here to come up with a double conjecture:

1. ${}^nC_{n-3} > n(n - 3)$ and 2. ${}^nC_{n-2} < n(n - 2)$, both for $n \geq 6$.

Let us take 1 first.

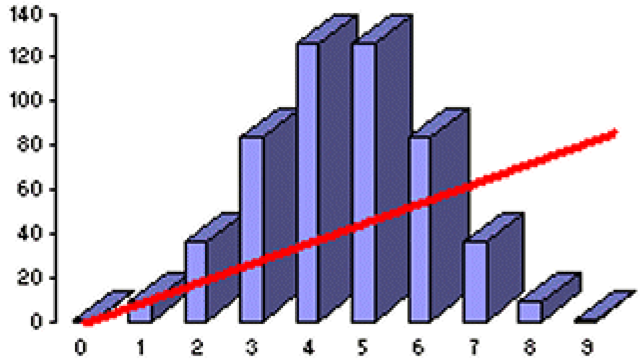
$${}^nC_{n-3} = n!/((n - 3)!3!) = n(n - 1)(n - 2)/6.$$

$$n(n - 1)(n - 2)/6 - n(n - 3) = n(n^2 - 9n + 20)/6 = n((n - 4.5)^2 - 0.25)/6 > 0 \text{ for } n \geq 6.$$

Now taking 2: ${}^nC_{n-2} = n(n - 1)/2$.

$$n(n - 2) - n(n - 1)/2 = n(n - 3)/2 > 0 \text{ for } n \geq 6.$$

It might help to think of this graphically. Below is the example ${}^9C_r$.

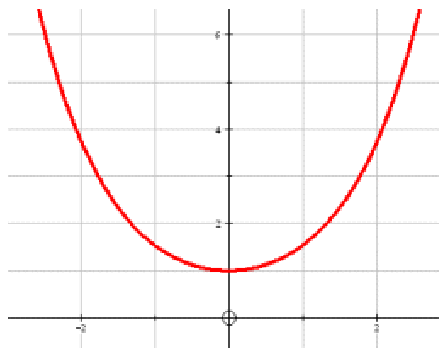


So will nr be closer to ${}^nC_{n-3}$ or ${}^nC_{n-2}$? ${}^nC_{n-3}$ increases as n^3 while nr and ${}^nC_{n-2}$ increase as n^2, so for $n \geq 10$, nr is closer to ${}^nC_{n-2}$.

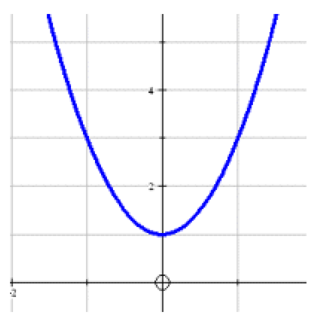
Risp 33: Almost Symmetrical

How close is a parabola to being a catenary?

Take the catenary y = cosh x.
(A chain suspended from two points will hang in a catenary.)



Try to fit a parabola to this curve.
(This could be the reflection of the path of a stone thrown through the air.)



Say the parabola has a minimum at (0,1), and it also goes through (a, cosh a).

What is the largest difference that there can be between the two curves?

What is the largest % error that there can be between the two curves?

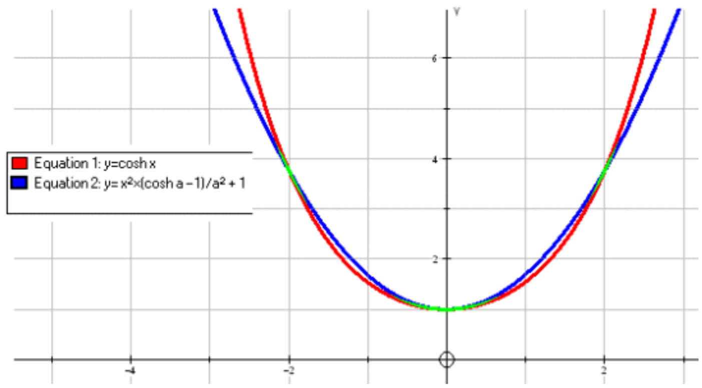
Are there any other curves that are quite close to being each other?

Risp 33: Almost Symmetrical

Teacher notes

Suggested Use: to consolidate/revise exponentials, percentage error, curve sketching

When asked what curve a chain takes up when left to hang naturally, most students will guess at a parabola. This is not far from the truth, but how far exactly? This risp invites students to try to put a number to it.



So the catenary has equation $y = \cosh x$.
If the parabola has equation $y = px^2 + 1$, and $(a, \cosh a)$ is on the curve, then $\cosh a = pa^2 + 1$, so $p = (\cosh a - 1)/a^2$.
So the equation of the parabola is $y = (\cosh a - 1)x^2/a^2 + 1$.

This means the difference between the curves at x is
$\cosh x - (\cosh a - 1)x^2/a^2 - 1$.

The dominant term here is e^x as x gets bigger.

The percentage error will be $100\% \times [\cosh x - (\cosh a - 1)x^2/a^2 - 1]/\cosh x$.

Here again $\cosh x$ dominates.

It is tempting to try to differentiate here, but it is quite a lot of hassle for not much reward. Sketching these curves in Autograph and seeing what happens as a varies is instructive and easier.

Risp 34: Graphing Tiles

You are given the following tiles, where c is a constant.

$\boxed{x}\ \boxed{y}\ \boxed{=}\ \boxed{c}\ \boxed{+}$

How many sensible equations can you make from some or all of these?
(No repeats!)

The tiles must lie on a straight line: $\boxed{x}\,^{\boxed{y}}$ **is not allowed.**

Enter your equations into Autograph.
Autograph gives c the value 1 to start with.

Now alter c using the Constant Controller.

Can you find a value for c so that:

1. Three of your curves are straight lines that enclose an equilateral triangle?
What is its area?

2. Two of your equations represent a curve and a straight line that touch?
Where do they touch?

3. Are there any lines of symmetry?

What other questions could we ask about this situation?

Risp 34: Graphing Tiles

Teacher notes

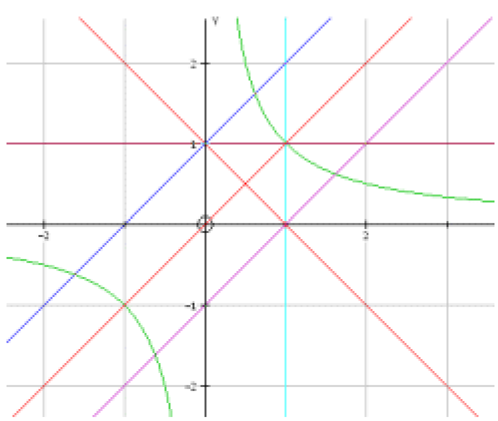
Suggested Use: to introduce/consolidate/revise curve-sketching, co-ordinate geometry, simultaneous equations

Reader, I have a confession to make. Ten months ago I promised you that all these risps would have been trialled in my classroom before being posted. However, in the last few weeks examinations have reared up, and what little time I have had with my students has been eaten away by the pressing matter of getting grades. Which is not to say that risps cannot be helpful in that quest, yet nevertheless, I find myself offering you a risp as yet unforged in the fire of the teacher-student encounter. Please accept my apologies, and let us hope a broken promise might be an opportunity, in a way.

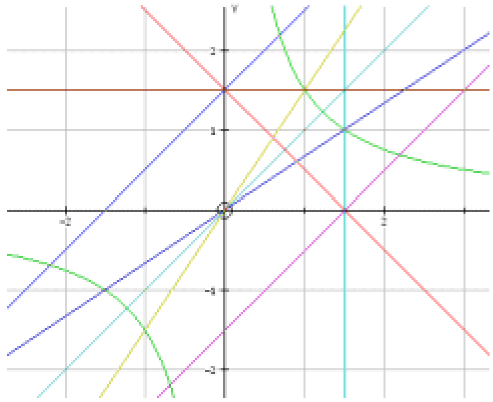
It turns out that using the given tiles, 9 reasonable equations fit for graphing are possible.

$x + y = c$
$x + c = y$
$y + c = x$
$xy = c$
$x = cy$
$y = cx$
$x = c$
$y = c$
$x = y$

Plotting these into Autograph gives the following initial picture:



Then increasing c slightly gives this:



Four of the nine curves are collinear only if c = 0.

Three of the lines enclose an equilateral triangle when c = tan 15°.
Its area? y = cx and x + y = c meet at (c/(1+c), c^2/(1+c)), so the length of side is 0.218779...
So the area of the triangle is 0.0207.
Another equilateral triangle is formed when c = tan 75°.

A line touches a curve when x + y = c touches xy = c, that is, when c = 4. The touching point is (2, 2). I would get students to observe this on the graph, then to check with algebra. If c = -4, we find the curve xy = c is touched by two lines, at (2, -2) and (-2, 2).

Risp 35: Index Triples

ϕ = 1.618033989...

e = 2.718281828...

π = 3.141592654...

How many different numbers can you make by replacing the squares below with ϕ, e and π in some order? (No repeats!)

$$\square^{\left(\square^{\square}\right)}$$

Without using a calculator, try to place these numbers in order of size.

If k = (largest of these numbers/smallest of these numbers), to which of these powers of 10 would you guess k is closest?

1, 10, 100, 1000, 10 000

Now check your answers with a calculator.

* * * * *

Can you find three distinct numbers a, b, c so that

$$a^{\left(b^{c}\right)} = c^{\left(a^{b}\right)} \text{ ?}$$

Can you find three distinct numbers a, b, c so that

$$a^{\left(b^{c}\right)} = c^{\left(a^{b}\right)} = b^{\left(c^{a}\right)} \text{ ?}$$

How many of the following numbers can be equal if a, b and c are all distinct?

$$a^{\left(b^{c}\right)} \quad c^{\left(a^{b}\right)} \quad b^{\left(c^{a}\right)}$$

$$a^{\left(c^{b}\right)} \quad c^{\left(b^{a}\right)} \quad b^{\left(a^{c}\right)}$$

Risp 35: Index Triples

Teacher notes

Suggested Use: to consolidate/revise ideas of curve-sketching, indices

Simple to state, this initial problem is one for which our previous mathematical education may have left us unprepared. Estimating the sizes of the six numbers proves remarkably difficult, yet we are left feeling that it is a fair question, one of which we should be able to make a reasonable fist. It turns out that:

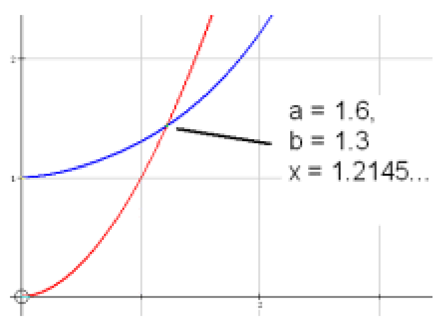
$$\pi^{(\phi^{e})} = 69,\quad e^{(\phi^{\pi})} = 93,\quad \pi^{(e^{\phi})} = 321$$

$$e^{(\pi^{\phi})} = 586,\quad \phi^{(\pi^{e})} = 49\,395$$

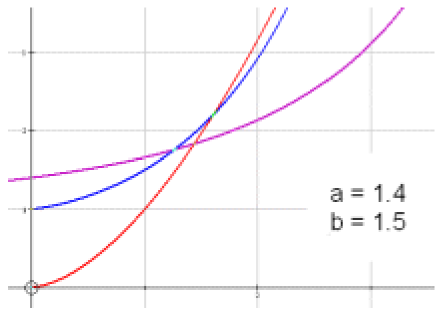
$$\phi^{(e^{\pi})} = 68\,567$$

So k is 1 000 (and how many people guessed that?)

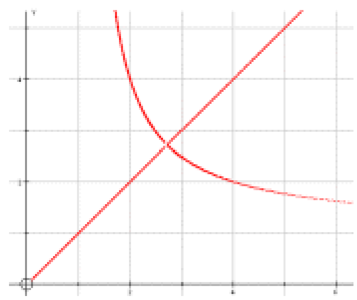
A graphing package is once again our friend for the later parts of the risp. We can plot y = x ^ (a ^ b) and y = b ^ (x ^ a) and find that there is no problem locating (x, a, b) triples where the two curves meet.



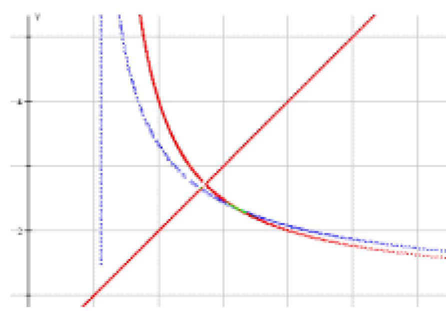
However, when we add the curve y = a ^ (b ^ x), the only time the three curves are coincident seems to be when two of the three numbers x, a and b are identical.



Moving to the final part of the risp, we can get three of these numbers to be equal. Let us aim for a ^ (x ^ y) = a ^ (y ^ x) = x ^ (a ^ y). For the first part of this we need x ^ y = y ^ x, and Autograph will draw us this curve implicitly.



If we now try to add the graph a ^ (x ^ y) = x ^ (a ^ y), we find that our graphing package may well baulk at this. However, (x ^ y) ln a = (a ^ y) ln x could be more possible.



The curves coincide at, for example, a = 1.1, x = 3.222..., y = 2.333...

We can then check that 1.1^(3.222^2.333) = 1.1^(2.333^3.222) = 3.222^(1.1^2.333)
= (approx) 4.309...

Note: the solution here is <u>not</u> a = 11/10, x = 29/9, y = 7/3.

Risp 36: First Steps into Differentiation

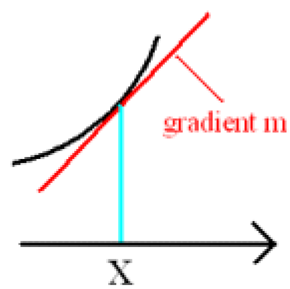
Open Autograph and draw the curve $y = x^2$.

Now draw $y = mx + c$.

(Initially Autograph will give both m and c the value 1, and will draw $y = x + 1$.)

Pick and enter a value for m, using the Constant Controller.

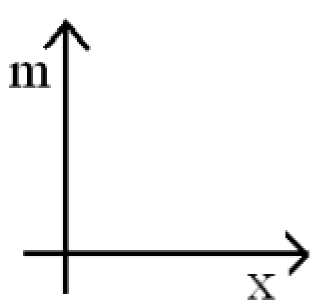
Now adjust c using the CC until $y = mx + c$ just touches the curve.



Make a table like the one below, and record your values for m and x.

m						
x						

Repeat this five more times, then plot m against x using the values in your table.



We call this 'the graph of the gradient function of x^2'.
What is the equation of this graph?

Now repeat the above, starting with $y = x^3$.

What do you get when you plot m against x this time?
What is the gradient function of x^3.

Suppose we start with $y = x^1$? What happens here?

Suppose we start with $y = x^0$? What happens here?

Put your results into the table below. Can you predict how the table will continue?

power of x	gradient function

Risp 36: First Steps into Differentiation

Teacher notes

Suggested Use: To introduce differentiation

One of the big questions that A Level maths teachers have to face is how to embark upon the Calculus. Should you start with the 'first principles' approach? After trying lots of things down the years, I have come to the conclusion that it is better to begin with something practical, something that can be easily visualised. Here I ask students to work with gradients using Autograph to do all the hard work. First principles can always come later, by which time the weaker students will be feeling that this calculus thing is okay. If they then find the rigour of first principles too much, they will retain their first impression that the rules they have discovered and can differentiate by are easy to use.

Here we arrive at the rule for differentiating a power of x through pattern-spotting, which is not a million miles from the technique that Newton used right at the start of calculus. I like the idea that as mathematics teachers we are trying to rerun the history of the subject inside our students' heads, only quicker! Maybe discovering the rule for differentiating x^n is such an important event in that history that this activity wins its place in an AS course where time is tight.

Risp 37: Parabolic Clues

You are asked to deduce the values of a, b and c for the parabola $y = ax^2 + bx + c$.

There are four clues to help you:

1. the y-intercept is (0, 6)

2. The curve goes through (4, 5)

3. the curve has a turning point at (2, 3)

4. the line of symmetry is x = 1.

It is not possible for all four clues to be true together. (Why not?)

Which combinations of clues enable you to find a set of values for a, b and c?

Find the resulting parabola for each successful combination.

Risp 37: Parabolic Clues

Teacher notes

Suggested Use: To introduce/consolidate/revise quadratics, curve-sketching

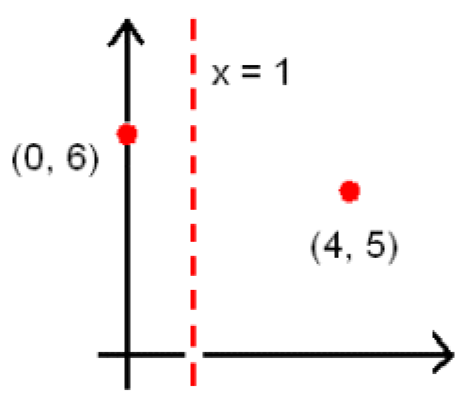
One of the best uses for a risp is to revisit material in a fresh way some time after the basics have been learnt. If you teach in a sixth form college as I do, the end of the summer term is an excellent time for this kind of reconstruction. Students are always in and out of lessons, with odd exams, trips and university open days, and pressing on with lots of new theory can be difficult with people missing. A risp activity builds links between the different parts of a student's existing understanding, and can helpfully reveal areas where there are gaps to be darned.

My students scratched away at this risp without making huge progress to start with. They could immediately see why clues 3 and 4 were incompatible, but they found it hard to come up with the winning combinations.

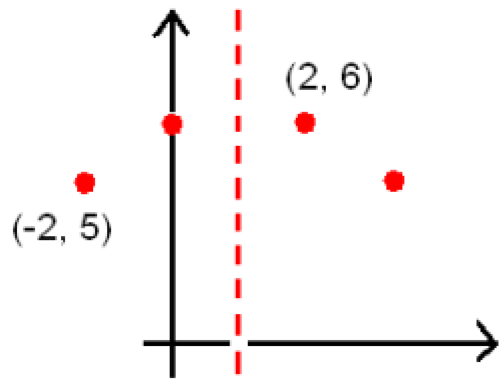
Does this explanation work? Two unknowns need two equations for a solution, three need three and so on. So we need three clues to find a, b and c. However, some of our clues are 'single' clues (1, 2 and 4) while 3 is a 'double' (we might see 3 as a Goes-Through-A-Point clue + a Line-Of-Symmetry-Is clue.) So that gives us as possible working combinations [1, 2, 4], [1, 3] and [2, 3]. [1, 2, 3] might be discussed, but it is contradictory.

It is amazing how many students try to embark on this without a diagram. Why is such a helpful thing ignored?

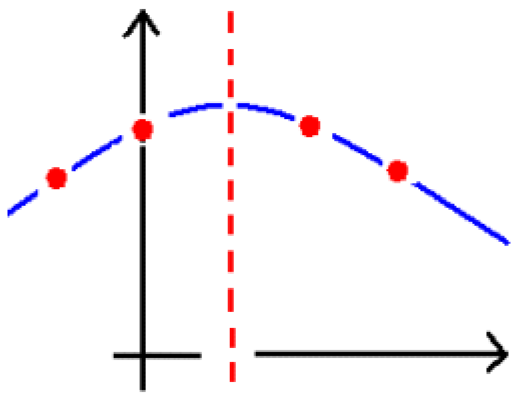
Let us start with [1, 2, 4]. Sketching what we know gives this:



The temptation is to insist that the parabola points downwards, in other words, to assume that a is positive. I had several students claim that 1, 2 and 4 together was 'impossible'. But eventually we agreed that symmetry means we have two more points on the curve, and they came round to this:



Now a parabola seems perfectly possible, with a being negative.



The equation of the parabola is $y = ax^2 + bx + 6$.
Putting (4, 5) and (2, 6) into this gives two equations for a and b, giving $a = -1/8$ and $b = 1/4$.

So $y = (-1/8)x^2 + (1/4)x + 6$.

Moving on to clue combination [1, 3]:

We know that $y = ax^2 + bx + 6$ again.
We know that (2, 3) and (4, 6) lie on the curve, so we use the method above to get:

$$y = (3/4)x^2 - 3x + 6.$$

However, there is the chance here to show off a good use of Completing the Square.
If (2, 3) is a turning point, then the equation of the parabola must be $y = a(x - 2)^2 + 3$. Now putting (0, 6) into this gives the required value for a.

The clue combination [2, 3] follows similarly.
We know $y = a(x - 2)^2 + 3$, so put (4, 5) into this, giving:

$$y = (1/2)x^2 - 2x + 5.$$

An alternative approach would be to recall that the line of symmetry of a parabola has the equation $x = -b/2a$. (This was not generally well remembered!)

I would guess that had my students been presented with this risp just as they were meeting quadratic functions for the first time, the less confident ones would have been scared off. By the start of their A2 work however, most had developed a mathematical maturity that enabled them to get a good deal out of the activity. Lots of interest was evident, and possible extensions to cubics and quartics were discussed.

Risp 38: Differentiation Rules OK

You are given the following six cards:

x	(x^2)	)
y =	(	exp

How many ways can you place them ALL into an equation that makes sense?
(No repeats and none left over!)

For each equation, find y' and the value of y' when x = 1.

How many different y'(1) values can you find?

Risp 38: Differentiation Rules OK

Teacher notes

Suggested Use: to consolidate/revise Product, Quotient, Chain Rules for Differentiation

As we move on to more advanced differentiation, we need to introduce our students to the Product Rule, the Quotient Rule and the Chain Rule. Essential to understanding which rule to apply in which situation is a grasp of which of the following bracketings apply:

$y = (\quad)(\quad)$ Product Rule

$y = \dfrac{(\quad)}{(\quad)}$ Quotient rule

$y = (\;(\quad)\;)$ Chain rule

This risp is designed to encourage such an understanding.
There are 10 equations that make sense:

1. $y = (x^2)(\exp x)$ *[Product Rule needed]*
2. $y = \exp((x^2)x)$ *[Chain Rule needed]*
3. $y = (x\exp(x^2))$ *[Product Rule, Chain Rule needed]*
4. $(x^2)y = \exp(x)$ *[Quotient Rule needed]*
5. $(x)y = \exp(x^2)$ *[Quotient Rule needed]*
6. $\exp(x^2)y = (x)$ *[Quotient Rule, Chain Rule needed]*
7. $(\exp x)y = (x^2)$ *[Quotient Rule needed]*
8. $\exp y = ((x^2)x)$ *[Differentiation of ln needed]*
9. $(x^2)\exp y = (x)$ *[Differentiation of ln needed]*
10. $(x)\exp y = (x^2)$ *[Differentiation of ln needed]*

The corresponding expressions for y' are:

1. $y' = x^2ex + 2xe^x$ *[y'(1) = 3e]*
2. $y' = 3x^2e^\wedge x^3$ *[y'(1) = 3e]*
3. $y' = e^\wedge x^2 + 2x^2e^\wedge x^2$ *[y'(1) = 3e]*
4. $y' = (x^2e^x - 2xe^x)/x^2$ *[y'(1) = -e]*
5. $y' = (2x^2e^\wedge x^2 - e^\wedge x^2)/x^2$ *[y'(1) = e]*
6. $y' = (1 - 2x^2)/e^\wedge x^2$ *[y'(1) = -1/e]*
7. $y' = (2xe^x - x^2e^x)/e^{2x}$ *[y'(1) = 1/e]*
8. $y' = 3/x$ *[y'(1) = 3]*
9. $y' = -1/x$ *[y'(1) = -1]*
10. $y' = 1/x$ *[y'(1) = 1]*

Thus there are 8 different values for y'(1) to be found.

Can anybody collect the whole set?

Risp 39: Polynomial Equations with Unit Coefficients

Pick an integer between 2 and 10, and call it n.

Write out the equation:

$$x^{n} = x^{n-1} + x^{n-2} + x^{n-3} \ldots + 1.$$

How many roots does it have?

Use a graphing package to find these roots fairly accurately.

Now explore the equation above for other integral values of n.

How many roots will the equation have in general?

How can we find these roots very accurately?

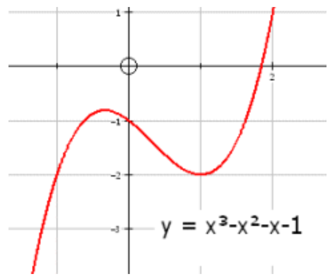
What happens as n tends to infinity?

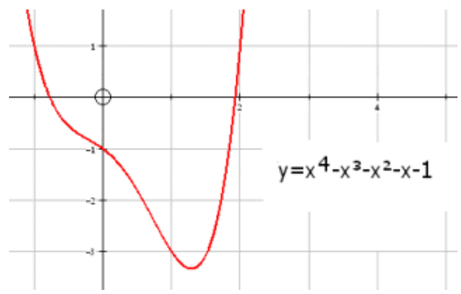
Risp 39: Polynomial Equations with Unit Coefficients

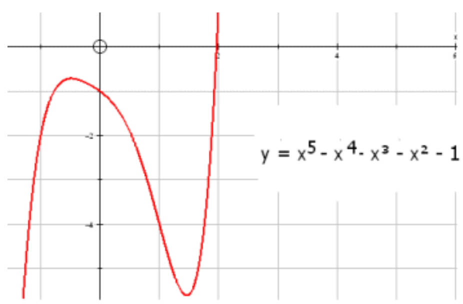
Teacher notes

Suggested Use: To introduce/consolidate/revise numerical methods, polynomial equations

This piece of polynomial-solving is a nice entry into the world of numerical methods and iterative procedures, although it will work just as well as a review exercise.







Playing with $y = x^n - x^{n-1} - x^{n-2}$ .. - 1 on a graphing program, a student will quickly develop the hypothesis that y = 0 has two roots (ever closer to 2 and -1 as n increases) if n is even, and one root (ever closer to 2 as n increases) if n is odd. We can find the roots accurately by homing in on the graph, but using an iterative procedure works well here.

Taking n = 3, we are trying to solve $x^3 = x^2 + x + 1$,
or $x = \sqrt[3]{(x^2 + x + 1)}$. This gives us the iteration $x_n = \sqrt[3]{(x^2_{n-1} + x_{n-1} + 1)}$,
which converges to 1.839... when $x_0 = 1$.

This same technique works for n even to find the root near 2.
$x_n = \sqrt[4]{(x^3_{n-1} + x^2_{n-1} + x_{n-1} + 1)}$ gives the root as 1.928...
With n even, can we find the root near -1 this way?
The fourth root is always positive (if it exists), so no.
The iteration above gives the root near 2 when x_0 is negative.
However, if we tweak the formula slightly as follows:
$x_n = \sqrt[3]{(x^4_{n-1} - x^2_{n-1} - x_{n-1} - 1)}$, we eventually get -0.7748 after entering $x_0 = -0.5$.

How to look at what happens as n tends to infinity?
Multiplying both sides of the following:

$$x^n = x^{n-1} + x^{n-2} + .. + 1$$

by x - 1 (with x not equal to 1), we get:

$$x^{n+1} - x^n = x^n - 1.$$

Thus $x^{n+1} = 2x^n - 1$

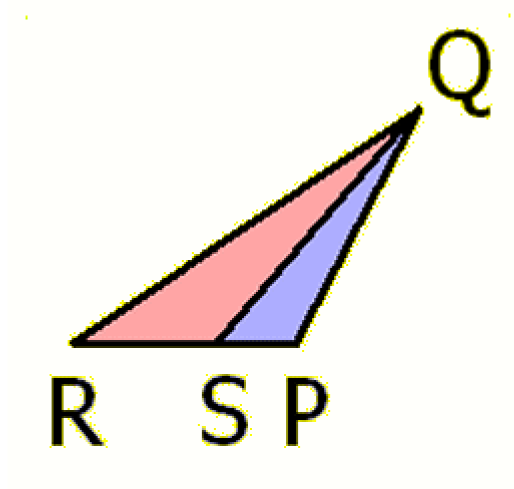
and $x = \sqrt[n+1]{(2x^n - 1)}$.

As n (even or odd) gets larger, if $x > 1$ then -1 becomes insignificant,
so we can see $x = 2$ is ever closer to being a root.

The -1 root is trickier to explain.
Let $(-1 + k)$ be the root of $x^{n+1} - 2x^n + 1 = 0$ where n is even.
Thus $(-1 + k)^{n+1} - 2(-1 + k)^n + 1 = 0$.
Neglecting k^2 and higher powers of k, we get:
$-1 + (n + 1)k -2 + 2nk + 1 = 0$, so $k = 2/(3n + 1)$ which tends to 0 as n tends to infinity.

The nice thing here is that this simple set of polynomials provides an excellent starting example for iterative techniques, an area where is it easy to embark on equations that behave in ways that we hoped they would not! Their form is hopefully easy enough to remember from one year to the next.

Risp 40: Perimeter Ratio



RS is 2 units, SP is 1 unit.

Q moves so that the perimeter of the red triangle is double that of the blue.

What is the equation of the locus of Q?
What does this curve look like?
What is the maximum value of RQ?

Now Q moves so that the perimeter of the red triangle is k times that of the blue.

What is the equation of the locus of Q?
What does this curve look like as k varies?
What is the maximum value of RQ?

Risp 40: Perimeter Ratio

Teacher notes

Suggested Use: to consolidate/revise co-ordinate geometry, curve-sketching

This last risp is perhaps the least syllabus-based of them all, a final bit of fun. It provides an entry point for a discussion about the equations of conics. To demonstrate that any bit of mathematics can be enriched (enrisped?) when viewed in the right way, I set myself the challenge of turning the Risp logo itself into a risp.

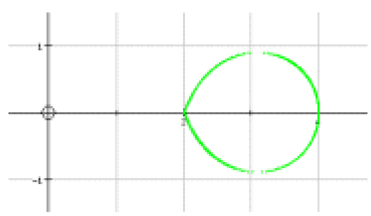
The best way into this problem is to set up Cartesian axes with the origin at R, with S at (2, 0) and with P at (3, 0). Then it is not too far to the following equation:

$$\sqrt{(x^2 + y^3)} + 2 + \sqrt{((x - 2)^2 + y^2)} = 2(\sqrt{((x - 2)^2 + y^2)} + 1 + \sqrt{((x - 3)^2 + y^2)}$$

This simplifies to:

$$\sqrt{(x^2 + y^3)} = \sqrt{((x - 2)^2 + y^2)} + 2\sqrt{((x - 3)^2 + y^2)}$$

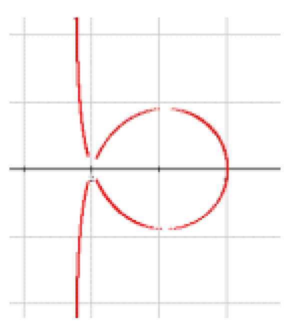
This can happily be graphed implicitly:



If we try to eliminate the square roots in this equation, we eventually arrive at:

$$4x^3 + 4xy^2 - 32x^2 - 7y^2 + 80x - 64 = 0.$$

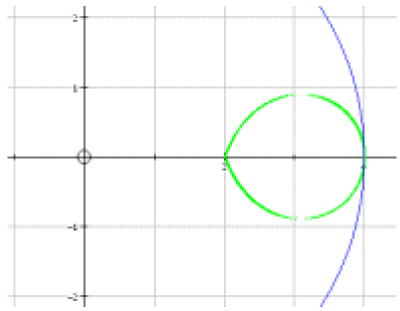
Graphing this gives the following:



The extra bits of curve arise through our squaring.

When is QR a maximum? Drawing $x^2 + y^2 = 4$ gives the answer:

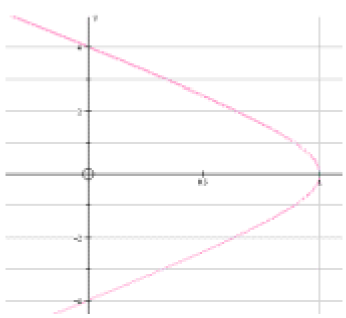
QR = 4 when y = 0.



When we replace the number 2 with k in the problem, we get the following equation:

$$\sqrt{}\,(x^2 + y^3) + 2 - k + (1 - k)\sqrt{((x - 2)^2 + y^2)} - k\sqrt{((x - 3)^2 + y^2)} = 0.$$

Trying to get rid of the square roots this time is <u>not</u> recommended! So what happens here as we vary k? Increasing k from 2 gives a curve of smaller and smaller area based around (3, 0). When k = 3, this curve disappears. If we decrease k from 2 towards 1, the area of the curve gets larger and larger, looking almost circular for a while before becoming reminiscent of a cardioid. Always, however, the largest value of QR happens where y = 0 on the curve. Putting y = 0 into $\sqrt{}\,(x^2 + y^3) + 2 - k + (1 - k)\sqrt{((x - 2)^2 + y^2)} - k\sqrt{((x - 3)^2 + y^2)} = 0$ gives (eventually) $x = 2k/(k - 1)$ as the maximum value for QR. Clearly something happens here when k = 1. Suddenly the curve appears to become unbounded.



If we try to find the curve's equation, we get:

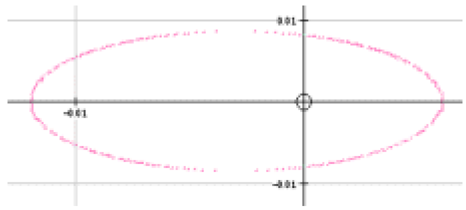
$$\sqrt{}\,(x^2 + y^3) + 1 + -\sqrt{((x - 3)^2 + y^2)} = 0$$

Removing the square roots (everything cancels obligingly) gives:

$$(x - 13/4)^2 - (y/2)^2 = 251/48.$$

So we have a hyperbola, and so QR can be as large as we wish if k = 1. (We only want one half of the hyperbola: the other half has been introduced again by squaring.)

Reducing k further, we arrive at the egg-shaped curve below, based around the origin.



How far can we reduce k? When k = 2/3, the curve disappears once more.

It is worth noting that the area of RSQ is <u>always</u> twice the area of SPQ.

These risps were originally posted to the Risps website at a rate of one a week over the academic year 2005-6. Below are Risps 1 to 40 listed in chronological order. While less useful to teachers planning to use the material, this list does tell a story: each risp built upon the previous ones, and the reaction to each risp affected subsequent choices each time. The later Teachers' Notes refer back to earlier risps, and some of the feedback that I received across the year is built in too.

List of Risps by Number

Part Two: Contents

Introduction

Rich: "pregnant with matter for laughter."

Starting: "making a sudden involuntary movement, as of surprise or becoming aware..."

Point: "that without which a joke is meaningless or ineffective."

or...

Rich: "productive, fertile."

Starting: "beginning, setting out."

Point: "place or station."

Chambers Twentieth Century Dictionary

Why "Rich Starting Points?"

This is something of a zeitgeist phrase -
(hopefully I've caught it just before it becomes a cliche.)
It is one with great cross-curricular potential:
this is what happens when you search the Internet for
"Rich Starting Point"...

Aeronautics
Climb-out test: From a rich starting point -
lean the engine until the full climb-out has the engine slightly accelerating the blades...

Theology
Furthermore, the act of being personal becomes a particularly rich starting point
towards the knowledge of God, fullness of the personal Being and...

Music
It is this sort of action, outwardly clear but made up of infinitely complex elements,
that provides such a rich starting point for much of the music herein...

Economics
A sustained bull market would have a tough time getting a foothold
from such a historically rich starting point...

Architecture
A spacious 1902 brick building provided a rich starting point
for this restaurant on Jackson Square which the jury recognized for its use of...

and similar examples can be found in the fields of Theatre, Art and Design, Photography, Literature, Genetics, Anthropology, Geography, Information Technology, Biology, Social work, Psychology, Accounting...

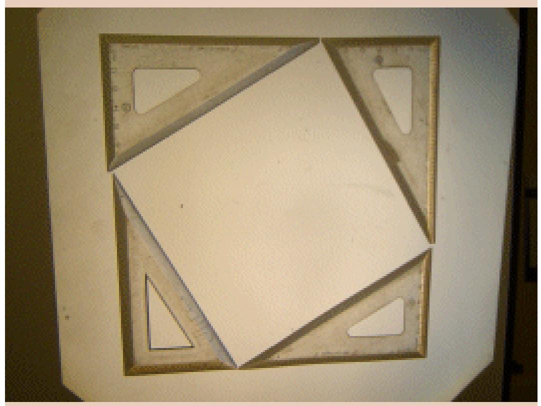
So for whom is this ebook intended?

Primarily for those who prepare 16-19-year-old mathematics students for A level.

If other teachers, or students, or teacher trainers,
or anybody else find this site useful, then I will be delighted,
but I am trying first and foremost to be useful to my fellow practitioners.

This site will judged on whether it directly or indirectly
touches real classrooms (including mine) in a positive way.

Uses for a Setsquare 1.



Pythagoras

Isn't this an old idea under a new acronym?

The idea that mathematics teaching should include, indeed, should be based upon, open-ended, investigatory, problem-solving activities is as old as the hills. But maybe the most obvious truths are sometimes the most easily forgotten. I hope I am not claiming to have re-invented the wheel here.

Perhaps some maths teachers have been less confident about using this type of work at A Level than at GCSE, falling back on more didactic methods for the 'advanced' material. My belief is that it is no harder to construct and use this kind of activity at A Level than at GCSE or Primary level, or indeed at University.

As for the acronym 'risp', I too have a love-hate relationship (that's an 'lohar') with acronyms. They are divisive: you either know what they mean (you're in) or you don't (you're out). But they can act as a useful gathering-point for peoples' thinking. I would invite you to live with the acronym 'risp' for now.

Doesn't the acronym 'risp' exist already?

Indeed it does:

Maybe there's room for one more...

How did the Risps website develop?

Risps - for A Level Mathematics was launched at the start of September 2005,

and the final risp was posted at the end of July 2006.

As I write, I plan to keep the site live for the foreseeable future.

Although no new risps will be added, the materials that are there will be refined and simplified as they are used more and more in the classroom.

My hope is that if you try out these risps you will give me some feedback,

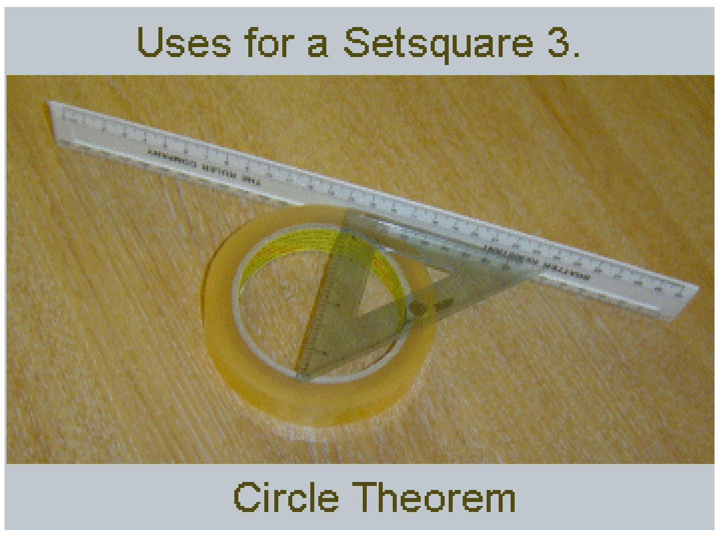
and that the site will evolve in response to your comments.

Hopefully too a methodology will be refined in the light of this dialogue.

Is there a risp for every A Level maths topic?

All the risps here tackle topics from AS/A2 Core (Pure) Mathematics.
Applied/Further Mathematics risps would be just as valuable
and these areas could become a future project.

(Almost) every risp I have posted here has been tried in my classroom
and refined in the light of this experience.



Additionally, every risp here leads directly into the A Level Mathematics syllabus.
I hope these risps will enrich lessons
without being tangential to the main business of the classroom.
If a risp does not practise key elements of the syllabus, it has not been included.

Are all of these risps original?

As far as I know, these risps, or at least the way they are developed,
are my creations.

But the same thing has been discovered by different people independently
so often in the history of mathematics, that it would be a surprise
if there was not an overlap with the thinking of other mathematics teachers here.

Maybe my unconscious will lead me to
forget to attribute an idea that I have picked up from someone else.

If you have a copyright worry about anything in this ebook,

then please contact me.

Risp Software Requirements

This is not a collection of activities that relies heavily on cutting edge technology.
Many of these risps require nothing more than a pencil and paper and some thought.
However, there are many that do rely on graphing software,
whether on desktop computers, on laptops or on graphics calculators.
In particular, several risps make use of the Constant Controller/Dynamic Constant facility
that is becoming a standard part of most graphing packages.
This enables you to enter an equation where the coefficients are variables,
for example, and to vary these simply.
Your package also needs to be able to graph implicitly defined functions,
and to draw the gradient function of a curve.

Two of the more popular graphing packages for PCs are Autograph and Omnigraph:
click below if you are interested in learning more.

http://www.chartwellyorke.com/autograph.html

http://www.spasoft.co.uk/omnigraph.html

On Macs, Gnuplot has been recommended to me. (It is free!)

http://www.gnuplot.info

If anyone knows of other helpful possibilities, please let me know.

Does using risps improve exam results?

Would you agree that it is possible to get good examination results by drilling students efficiently without building much long-term understanding? I think that in such a case knowledge will not be retained for long, and no-one is likely to have enjoyed the process much.

Can students attain a deep and lasting understanding and fail to do well in exams?

I guess if the preparation for the examination itself is really disorganised, then this might be possible. But surely a relaxed and flexible familarity with the concepts and how they are linked together should normally carry over into any exam.

I am sure that using risps can lead a confidence and fearlessness over mathematics, that leads in turn to the kind of examination performance that every good classroom teacher must insist upon.

Is everything in this ebook true?

Everything here that is not meant to be a joke is true.

Unless it makes for a better story,
in which case that might not be true either...

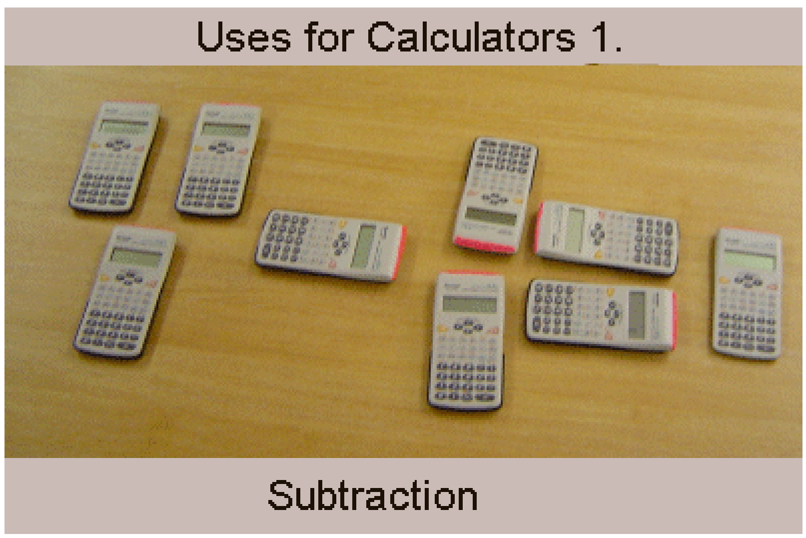
I believe Freud was once described as a lousy scientist but a great storyteller.

Maybe the same is true of me.

Can I use this material as I like?

These risps are deliberately presented in a no-bells-and-whistles way.
The intention is that you should take the kernel of each risp idea,
make it your own, and customise it for your own situation.
Only you know your learners and the context in which you teach them.

So please, be as free as you like in your classroom with what is here.



If you wish to publish material from this ebook, please contact me.

The legal bit is as follows:

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Risp Methodology

Types of Risp

Broadly speaking, a risp may be used in three ways:

to introduce a topic

to consolidate a topic

to revise a topic, or a number of topics

An introductory risp is specifically designed to give a pathway into new theory. The hope is that by starting to ask for themselves some of the questions that the theory is intended to answer, students will be prepared for the exposition that is to follow.

A consolidation risp provides good practice in the use of skills while attempting to answer some wider question. This type of risp will assume some relevant theory to have been previously studied. A consolidation risp will generally be a good revision risp as well.

A revision risp is akin to a consolidation risp, but is likely to be more synoptic, drawing together bits of theory from a range of topic areas. This means that careful consideration needs to be given over when to use such a risp: are all the required skills in place?

The Teacher Notes always begin with a suggestion about the most appropriate use for each risp. You are of course free to disregard this suggestion (!)

How to Write/Choose a Risp

What follows are my tentative suggestions:

- Everyone should be able to see the point of the initial question, and be able to make a start, regardless of their prior technical knowledge.

- An element of student choice at the beginning of the exploration may be helpful: picking a number, or a curve, or an equation. Students will then take ownership of their example, and will it to succeed.

- It should be possible to extend the risp, so that all students can stay interested in the situation. Weaker students should find the risp generates accessible and engaging material throughout, while stronger students should be able to ask harder and deeper questions as they get into the problem. In other words, differentiated learning starts to take place.

- A risp should encourage fresh questions, divergent approaches, and open-ended thinking. A helpful risp will practice key mathematical skills indirectly, that is, the key skills are needed to tackle the risp and are not simply practiced for their own sake. A good risp will stimulate curiosity in the student: "What if I do this?"

- A good risp is often synoptic, calling on the syllabus in its entirity, and rehearses insights and techniques that the student will not have practiced for a while. This encourage the formation of links between different parts of mathematics, which is a great aid to understanding.

- It will often be the case that a risp will find ICT helpful in some way, often as a labour-saving device.

How might a Risp fit into a Lesson?

Here is how I tend to use risps in my teaching currently.
I offer this more as a springboard for the imagination than as a strict formula.

- I am a huge fan of the overhead projector.

- I explain to my students at the start of the year that I want them to be prepared to think hard about a fresh problem at the start of each lesson. I won't give them much initial help with these: they must be ready to be creative self-starters from the word go.

- My students often come in to my classroom to find a risp on the OHP already. The risp needs to be clear enough to enable students to get going straight away. I will give some words of welcome and make sure that everyone is getting their mind into gear.

- After the initial administration, I start by visiting those I currently consider to be the the weaker students. Are there words they do not understand? Are there misconceptions emerging? (If so, good!) Can they see the point of the question? (If not, the risp needs to be refined.) Ideally the risp creates an environment where students can freely discuss what they do and do not understand without feeling threatened.

- Now it is time for those I currently see as the the stronger students. By now they should have negotiated the start of the risp with relative ease. Now is the time to challenge them with the tougher end of the risp, and to ask them to go beyond it.

- Let me assume you have a full range of 'ability' in your class. If you divide the group into those you see as likely to get A-B grades, those likely to get C-D grades and those likely to get E-U grades, a rough rule of thumb is this:

The first third of the risp should be immediately accessible to all three groups.

The second third of the risp should be immediately accessible to the top two groups, and moderately accessible to the third.

The third third of the risp should be immediately accessible to the top group, and moderately accessible to the second.

- Of course, I am not saying that the membership of these groups is fixed and unchanging. Which students you consider to be in which group may change topic by topic and as the year progresses. Indeed, the beauty of teaching with risps is that if you underestimate a student, it matters not: they can show you what they are capable of doing by working on the risp alone. You do not need to change the task.

What might happen after the Risp?

- I end work on the risp to have a plenary, gathering together what has been discovered in a whole class setting. I would note that this is not a lecture, which is guaranteed to kill off student interest, but rather a joint construction of the theory that emerges from the risp. I am always aware of the need to improvise here. Although I have in mind the theory I wish to head towards, each class requires me to construct a different route to my goal. I could invite a student to address the class at this point.

- I tend to do the note-taking on the OHP while the students watch as they contribute. Asking the students to make comprehensive notes themselves as we do this gives the illusion of useful work, but while they are note-taking, they cannot really think. Sometimes looking afterwards at notes students have taken, I am shocked by how two or three slight errors can render the notes near to useless. Mathematics is so concise a language that this can often happen.

- I can now introduce related new theory that the risp does not immediately address. I find that students quite happily accept this part of the work with great concentration: the fact that they have asked to think independently at the start of the lesson somehow legitimates this more teacher-centred part of the lesson.

- I restrict myself to four A4 acetates worth of notes in total. I then get these reduced onto a single double-sided page of A4 and copied off for the students in time for their next lesson with me. Thus each student is given a one-page summary of the theory, to which they have contributed, that is unique to that particular class and that particular lesson. They then need to engage with these notes, by underlining, highlighting, linking and so on, to make them their own.

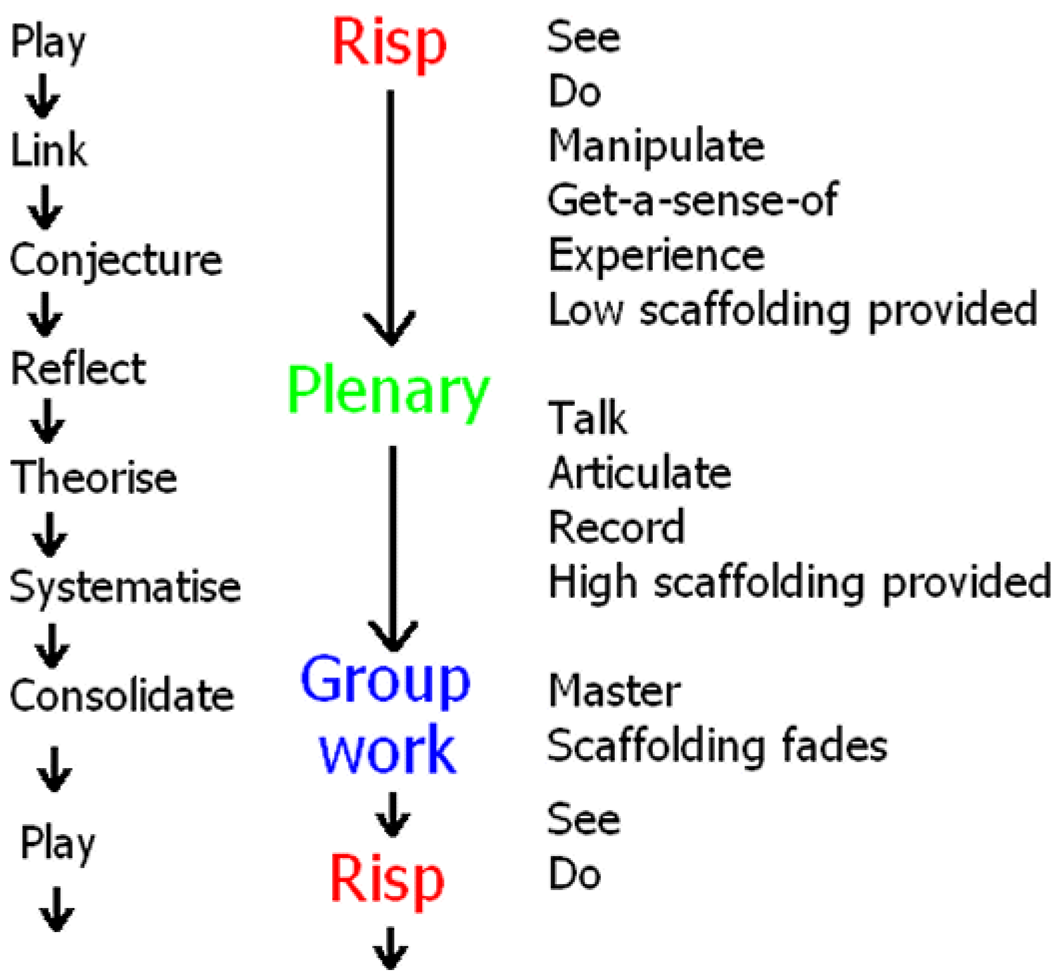
- Now it is time to consolidate the theory of the lesson. This does not have to mean exercises from books: the kind of group work, often involving games, that the Standards Unit is currently promoting would be ideal.

How much time would you give to a Risp?

- For a 90 minute lesson, maybe 30 minutes of risp, a 30 minutes of theory, and 30 minutes of consolidation. But of course, every lesson is different.

Some teachers might say, "I would love to put risps into my lessons, but I just don't have the time." My response would be that a carefully-prepared risp will save you time, because it will mean your students will understand things quicker, which means less time spent on consolidation and revision.

The following diagram might help:



This brings together a number of constructs from the literature.

'Scaffolding' (Bruner, Wood and Ross, 1976) - the act of being there as a student engages with a problem, and then allowing the student to build a structure for themselves with support, and then fading into the background once the structure is in place.

'See-experience-master' (Floyd et al., 1981) - initial seeing by the student tends to be superficial, growing to a more engaged experience before moving to a mastery where the details become of less importance.

'Manipulating-Getting-a-sense of-Articulating' (Mason, Johnston-Wilder, 2004) - when beginning a task there will be a manipulation of objects, that could be physical, symbolic or mental. As the investigation of the task continues, a student finds that underlying patterns and relationships are beginning to come into focus. When they truly own the task, they will be able to articulate what they have assimilated.

"Would you sow mathematics without ploughing first?"
Can you explain this Risp Workshop title?

Teaching is often compared in my mind with farming. There is the rhythm of the agricultural year that matches the rhythm of the educational year: our annual exam results roughly equate with a harvest. But our lessons need to have a rhythm too - ploughing, sowing, growth and reaping. What place does a risp have in the analogy? No farmer would attempt to sow without ploughing first. A farmer who does such a thing is akin to the teacher who launches straight into unmotivated theory at the start of a lesson, only to experience their students as stony ground. Using a risp prepares the seed-bed, stirring up the group and inducing a receptivity. Students take a while to get used to this, but when they do, I believe the benefits can be tremendous.

Using a Risp: What Can Go Wrong...

1. "It's too difficult!"

I guess the most common problem I run into with a risp is having immediate expectations of my students that are too high. This delicate balance between believing your students can achieve great things in mathematics one day, and knowing what it is realistic to expect of them in this particular lesson on this particular day is well expressed in the quotation below.

The most important feature of a task is that it contains some challenge without being overly taxing. That is, it does not appear to be trivial, nor does it appear to be beyond the learners' capabilities. Tasks which demonstrate to learners that they are capable of more than they can imagine can do wonders for building their confidence; on the other hand, tasks that have been 'suitably simplified' to be well within the learners' capabilities are likely to be seen by them as not worthwhile, and so may demotivate rather than inspire. A great deal depends on the atmosphere of trust between the learners and the teacher.

John Mason and Sue Johnston-Wilder, Designing and Using Mathematical Tasks, Open University, 2004

In my experience, students are far more often likely to complain a task has been too hard than too easy!

2. "I love it when my students are investigating for themselves..."

One of our feeder schools once had a maths teacher who taught everything through investigations. He was popular, the students loved his lessons, but the results at the end of the year were terrible. Is it not possible for a teacher to become carried away as his class descends into a risp? Is it not possible for this engrossment to become an end in itself, overriding all concerns of the syllabus, the examination and the Scheme of Work? It is intoxicating when your students get caught up in a risp: might this not be a danger sometimes?

The phrase 'teacher lust' has been coined to describe how a teacher may feel an overwhelming and unhelpful desire to explain everything for and to his students. Is there not such a thing as 'risp lust', where a teacher is so keen to allow his students to explore mathematics for themselves that they never really form a systematic collection of knowledge in their heads?

Yes, I believe that the spirit of investigation should be alive and well in every mathematics classroom. But the results of that investigation need to be carefully built into lasting structures, or else our students will do nothing each lesson but make patterns on the sand, that are washed away each night by the tide.

How many rich tasks should a teacher be expected to do in a course? Probably more than most do now. Probably fewer than most will likely want to do in the future.

A Handbook on Rich Learning Tasks, 2001,
Gary Flewelling with William Higginson (see Risp Books)

3. Risps Presentation

One major theme in this ebook is how a slight tweak in approach can bring a question to life. Most commonly the shift from a closed question to an open question will accomplish this. But even open questions need to be presented in the right way. It is that final sprinkling of parmesan and black pepper that makes the dish that little bit more exciting and which draws one in.

Take Risp 1. The original version of this asked:

"When is the difference between two triangle numbers prime?"

I guess as stated this is rather a closed question. It is also a little unwelcoming for students. The word 'prime' has to be attached to 'difference' which is a whole five words away. The phrase 'prime number' will be a lot more accessible to students than the adjective 'prime' on its own. They also need to recall the meaning of 'triangle number' at the same time as everything else. The question is too dense and a little off-putting.

After working through this with a couple of classes, the risp became this:

Pick two whole numbers between 1 and 10 inclusive, call them a and b.

Say that T_n is the nth triangle number.
Find T_a and T_b.

What is the difference between T_a and T_b?
Is this a prime number?

When is the difference between two triangle numbers a prime number?

When is the difference between two square numbers a prime number?
How about the difference between two cube numbers?

The first line is now a task that any student can accomplish. Every student experiences a feeling of partial success on this problem that no-one can take away!

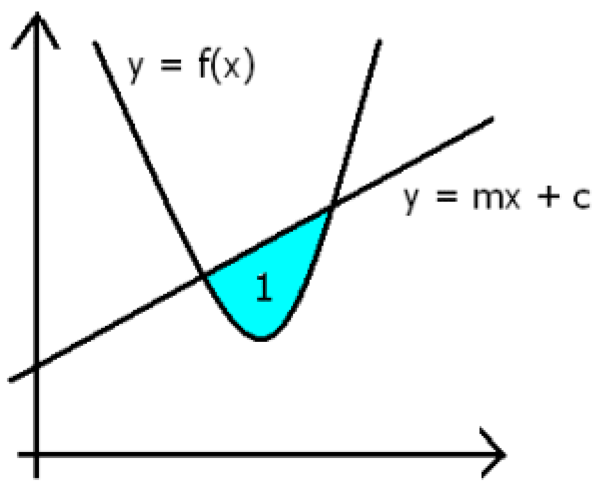
The second line leads students gently into algebra, while testing their understanding of 'triangle number'. The next line asks for diagrams, which must be a good idea. Each step is essential and yet accessible too. Then comes the revision of the idea of a prime number, at which point everything is in place to make sense of the main question. This can then be opened out in a variety of ways.

There are many times when I have come close to discarding a risp, only to discover that a slightly different way into the question can revitalise it.

4. "The algebra is getting horrible here..."

The textbook exercises that our students have grown up on are generally 'fixed' so that the numbers 'work out nicely'. This is not quite what happens in real life with mathematics, but it does have the advantage that students can concentrate on the skill they are meant to be learning without grappling with expressions that do not factorise or with surds that refuse to simplify happily.

However, when we ask our students to take on a risp, there are no such guarantees of numerical serendipity. They may well be choosing their own starting numbers, and given the openness of the problem, they may head off in all sorts of unexpected directions. A risp may have a simple solution with judicious early choices, but if students bark up Tree B rather than Tree A (the one that seems obvious to us), they may give themselves an algebraic headache. Risp 25 is a case in point:



So what is the answer? I would say that the onus is on the risp-writer to ensure that the weaker students in particular are given a carefully-structured opening to the risp that ensures a fairly safe passage through the initial moments. Maybe the choice of numbers or functions will have to be restricted at the start. In other words, effective early scaffolding needs to be in place.

5. "I am not confident with the maths on this one..."

We are told endlessly that the nation is short of maths teachers. There have been some laudable efforts to address this fact, notably the MEI Teaching Advanced Mathematics project, where maths teachers who have not taught beyond GCSE before are given the skills and knowledge to take the step up to teaching A Level. Now it is certainly true that the traditional Theory-Consolidation lesson is 'safer' for a fledgling A Level teacher than the Risp-Theory-Consolidation kind of lesson. Should someone whose background in mathematics is less extensive than that of a full mathematics degree take a risk over the improvisation a risp involves?

I guess some risps are riskier than others! It is hard for anyone to teach a risp 'blind'. I certainly think that a good 'traditional' lesson is better than a bad 'risp' lesson. If the risp goes off the rails somehow and the teacher does not have the experience to get it back on track, the class can be left feeling confused and unhappy.

On the other hand, a series of lessons where nothing unexpected ever happens will surely bore your students to death eventually. There must be an element of improvisation in everybody's teaching to keep the educational process alive. I think it is perfectly in order to say to students, "Now I should tell you I have never done this activity with a group before." They will feel a little special and be forgiving if there is the occasional hiccup.

I can truly say I have never come out of a risp-lesson seeing the activity in the same light. Students always suggest different facets to the task. This means I go into the lesson curious, hungry to see what will be suggested. And if I am feeling that way, might that not be infectious?

Risp Books

- ***Starting Points,* C.S.Banwell. K.D. Saunders, D.S. Tahta**
  Simply the best mathematics education book I've ever read. Witty, wise and hands-on. Where this site began.
  Tarquin 1986 (or earlier?) ISBN 0906212510

- ***Whatever Next?*** Ideas for use on A Level Mathematics Courses,
  An ATM activity book, written by a number of teachers.
  Lots of excellent ideas, helpfully presented. Encouraged me to experiment for myself.
  ATM 1988, ISBN 0900095725

- ***6th Dimension, Developing Teaching Styles in A Level Mathematics,***
  Editor: Laurinda Brown
  A wealth of rich and open possibilities for inquisitive mathematicians.
  Resources for Learning Development Unit, Bristol, 1986-1988, no ISBN

- ***Proofs and Refutations,* Imre Lakatos**
  Effectively argues for a risp-based approach to mathematics, rather than a deductivist one. Some really tough material at times here, but the start is fun and compelling.
  Cambridge University Press, ISBN 0521290384

- ***Designing and Using Mathematical Tasks,* John Mason and Sue Johnston-Wilder**
  Written for the ME825 course *Researching Mathematical Learning,* run by the Open University.
  Provides a wonderful philosophical framework for the construction of classroom tasks. Especially strong on a wide range of techniques for presenting any given risp.
  The Open University 2004, ISBN 074925534x

- ***Improving Learning in Mathematics: Challenges and Strategies,* Malcolm Swan**
  Outlines the philosophy underpinning the Standards Unit material for teaching A Level mathematics released in 2005. A range of ways to breathe life into your classroom.
  DFES 2005, Crown Copyright, ISBN 184478537x

- ***A Handbook on Rich Learning Tasks,* Gary Flewelling with William Higginson**
  This book is full of pithy philosophy concerning the use of risps in mathematics teaching. Especially strong on assessing rich tasks. Written in a warm and accessible style.
  Queens University, Ontario, 2001, ISBN - none
  Available from the AAMT (Australian Association of Mathematics Teachers)

Please feed back

This project has benefited hugely from constructive comments
both critical and affirmative in its short lifetime.
If you have tried a risp and would like to say how it went,
or if you have some wider thoughts about risps and their philosophy,
then I would be delighted to hear from you.

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With thanks to…

My most immediate debt of thanks is to Bernard Murphy, my main Gatsby Mentor for my Fellowship year, who took the trouble to visit my classroom twice in that time. Bernard also worked each risp thoroughly, giving valuable suggestions, and he was always on hand with encouragement at every stage. My further thanks goes to Susan Wall, my other Fellowship mentor, who gave me especially good advice over using these risps with weaker students. Lisa Berelian-Page, Sue Robinson and Charlie Stripp interviewed me right at the start of this project, and I am grateful to them for their warm support throughout.

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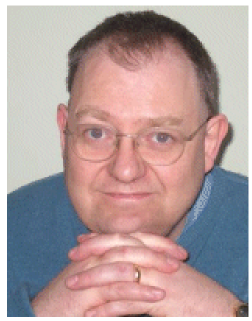
Jonny Griffiths, March 2007

About the Author

Jonny Griffiths teaches mathematics at Paston Sixth Form College in Norfolk, where he has been for the last fifteen years. Before Paston, he taught A Level Mathematics at St Dominic's Sixth Form College in Harrow-on-the-Hill, and at the Islington Sixth Form Centre. He also worked for eighteen months at St Philip Howard 11-16 comprehensive school in Tower Hamlets. He has studied mathematics and education at Cambridge University and with the Open University. Possible claims to fame include being a member of *Harvey and the Wallbangers*, a popular band in the Eighties, and playing the character *Stringfellow* on the childrens' television programme *Playdays*.



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